A Middle Path

Between Just Intonation and the Equal Temperaments

Part 1

8/27/04, updated 12/28/15 with corrected figures for Augene

Paul Erlich (e-mail: perlich@aya.yale.edu)

Many books and articles that discuss musical tunings and scales, whether in the context of history, theory, or new music, frame the issue as a dichotomy between just intonation and equal temperament. Even when the equal temperament concept is generalized, so that the octave may be divided into a number other than the conventional 12 steps, or the tuning made slightly irregular (as in "well-temperament"), the dichotomy remains woefully inadequate. Just Intonation usually refers to a system with at least 3 dimensions – a set of at least three basic intervals (such as the prime intervals 2:1, 3:1, 5:1, or alternately the intervals 9:8, 10:9, 15:16) is needed in order to be able to construct any interval of the system. Meanwhile, an equal temperament can be understood as a 1-dimensional tuning system, each of its intervals being constructible by repeatedly stacking the smallest step.

More important than either of these to Western musical history[i] and notation, however, has been meantone temperament, which is 2-dimensional (its basic intervals can be taken to be the octave and fifth, or the major second and minor second, for example). While recent decades have seen a proliferation of both just intonations and unconventional equal temperaments in both theory and practice, systems of intermediate dimensionality have gone virtually ignored. This paper and its sequel help to remedy this deficiency, by presenting a variety of 2-dimensional temperaments. Each temperament implies a series of scales, each scale a subset of the next in the series – for meantone, a segment of this series consists of the pentatonic, diatonic, and chromatic scales. These series of scales are depicted in a manner similar to the horagrams of Ervin Wilson. This paper also introduces a new criterion for optimally tuning a temperament, in which octaves may be tempered, and the temperaments and scales herein are specified in optimal form according to this criterion.

Foreword

This paper springs from a conversation of many years involving Graham Breed, David Keenan, Gene Ward Smith, and many other members of the internet's tuning and tuning-math groups.[ii] It scratches the surface of one aspect of this collaboration: the great variety of distinct classes of temperament that we have discovered (or often, rediscovered), composed, and performed[iii] in over the years. Also important is the mathematical theory that underlies, unifies, and structures this variety.[iv] In this paper, though, I have chosen to avoid all but the simplest mathematical concepts.

Introduction

Ancient musicians in China, Greece, and elsewhere attempted to relate their musical intervals with the lengths of string or pipe used to produce them. The ancients discerned the musical identities of many simple-integer string-length and pipe-length ratios (such as 2:1 – the octave, and 4:3 – the fourth) and deemed many of those ratios ideal in terms of tuning. In the sixteenth century, Benedetti became the first to relate the sensations of pitch to vibration frequencies. He equated the previously identified musical ratios with frequency ratios.[v] The centuries since have done little to invalidate that equation. And many continue to extol the ideal, exact simple-integer ratios between the frequencies of notes concordant[vi] with one another. Such ratios and the tuning systems built from them are referred to collectively as *JI* (just intonation).

Unlike JI, temperament has been a central feature in both the practice and theory of Western music for half a millennium. What is temperament? The word temperament refers to alteration. Musical temperament is the alteration of the exact simple-integer ratios at the heart of JI. The alteration is not arbitrary; it is designed to achieve certain goals.

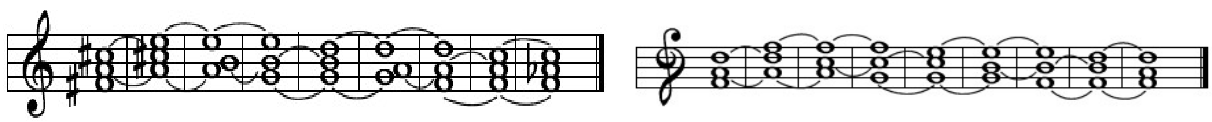
Historically, temperament allowed simple sets of notes, such as the familiar diatonic scale, to be tuned so that there were no *wolves* -- unusually discordant realizations of the intervals used as consonances.[vii] The pure or just tuning of the major triad has exact frequency proportions 4:5:6. It is well-known that the white-key or all-naturals diatonic scale, if tuned so that C major (C-E-G), F major (F-A-C), and G major (G-B-D) triads are just, will have discordant, out-of-tune intervals from D to F, and worse, from D to A. The ratios of D to F and D to A are exactly 32:27 and 40:27, respectively. These ratios are too complex to be heard as pure concordances. Proponents of just intonation have suggested introducing an additional version of D, a syntonic comma (81:80 or 21.5 cents[viii]) lower, for use specifically with F and A. This allows the two intervals in question to be realized as 6:5 and 3:2, respectively. The number of these additional pitches needed increases rapidly in schemes that go beyond a single diatonic scale and seek to accommodate more and more keys. But such proposals have generally had little impact on musicians and instrument makers due to their intricacy and departure from engrained habits.[ix] They also leave unresolved the question of how to tune chords such as C-E-G-A-D and F-A-C-D-G. Historically, temperaments slightly altered some or all of the concordant intervals so that the two versions of D would coincide exactly, eliminating these difficulties.

Thus temperament can increase the number of concordant sonorities possible with a given number of fixed pitches. Temperament, when implemented in a regular manner, also reduces the often bewildering variety of interval sizes in a scale or finite tuning system to a manageable few. This simplification is a great aid in notation, possibly cognition, and most famously, modulation (the transposition of patterns of notes from one pitch level to another). The reduction in the number of distinct pitches needed to explore wide harmonic areas is often quite substantial as well. This allows for more practical realization on a physical instrument (such as a keyboard or guitar) for composition or performance.

It is often argued that the issue of temperament is only relevant for instruments of fixed pitch, such as keyboards and fretted string instruments, while it's irrelevant for music involving instruments of variable pitch, such as the human voice. It is true that precise fixed-pitch tuning specifications assume both too much accuracy and too little flexibility to be relevant to the practice of vocal, string, or even wind ensembles. However, the entire corpus of common-practice Western music is written using notation that is compatible with only those classes of temperament associated with the diatonic scale, and roughly since Beethoven,[x] the closed system of 12 notes per octave. All the harmonic progressions and melodic resources in these compositions, regardless of instrumentation, conform to the structure imposed by the keyboard.

Diatonic progressions like C major → A minor[xi] → D minor → G major → C major, though they would necessarily involve pitch shifts or drift of a syntonic comma in strict just intonation, are by no means avoided in music for flexible-pitch instruments. In fact they're rarely absent.

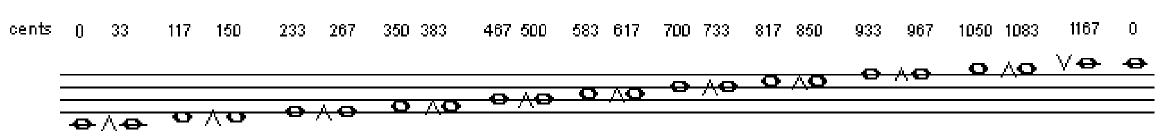
Meanwhile, many other simple harmonic cycles, which would necessitate shifts or drift in JI by other ratios, are possible. A few of these have been explored in the West since Beethoven – primarily in connection with enharmonicism in the 19th century and symmetrical scales, such as the diminished or octatonic scale, in the 20th. But most have been neglected in Western music, due to their incompatibility with any prevailing temperament. Here is a chord progression which, if realized in 12-equal (notated on the left), falls a semitone from the first chord to the last:



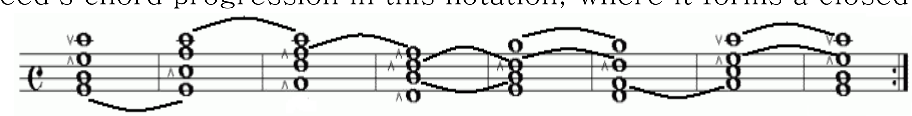
If realized in strict JI, with common tones held steady, the progression falls by the ratio 250:243. But in a temperament where 250:243 becomes a unison, the progression returns to its starting point, and its notes form an even 8-note scale (with 1 small and 7 large steps). Using this scale as the basis for a new staff notation, where the top and bottom lines are now an octave apart, the progression can be notated as on the right.

In the progression below by Graham Breed, most of the four-voice chords are meant to approach (aside from inversion) the frequency proportions 4:5:6:7; the rest, 1/7:1/6:1/5:1/4. If realized in 12-equal, this progression would drift up by a semitone:

In strict JI, though, the drift is only 2401:2400, an interval of less than 1 cent. Clearly a performance of this chord cycle by a flexibly-pitched ensemble, holding closely to just intonation ideals, is conceivable. Unfortunately, conventional notation is utterly incapable of conveying it. But a notation system based on a 10-note scale, rather than the diatonic one, is well-suited to the task. Accordingly, Breed suggested a "decimal" notation system;[xii] Joe Monzo implemented the suggestion on a four-line staff and created this diagram showing its representation of the "Blackjack" or "Miracle-21" scale in 72-equal (a tuning where 2401:2400, as well as 225:224, does indeed vanish):



Below is Breed's chord progression in this notation, where it forms a closed cycle:



The purpose of this paper is to bring to light a host of alternative temperaments (alongside a few familiar ones), and the scales which would be natural to notate music written in them. These should not be understood merely as lists of pitches to be employed when tuning an acoustical or electronic instrument. More importantly, they should be seen as models for the conception and notation of new music, regardless of the instruments or precise tuning strategies employed in its implementation.

The Pythagorean Lattice

To prepare us for considering temperament in more detail, we begin with an infinite just intonation (JI) system.[xiii] Typically, this system is defined by specifying a *prime limit*. The prime limit is the largest prime factor[xiv] to appear in any of the ratios. The lowest possible prime limit is two, but this only allows for a single note and its octave transpositions. JI with a prime limit of three is often referred to as Pythagorean tuning. The tuning system can be represented geometrically as a lattice.

Figure 1 shows a portion of the three-limit lattice, with notes represented as circles. Within each circle is the corresponding note's name – the modern nomenclature used here originated in the medieval era, when Pythagorean tuning was the norm. The subscript, indicating octave register, is a more recent development. Figure 2 denotes the same notes with the ratios of their frequencies to that of an arbitrary tonic (corresponding to C_4 in Figure 1). Both figures courtesy of Dave Keenan.

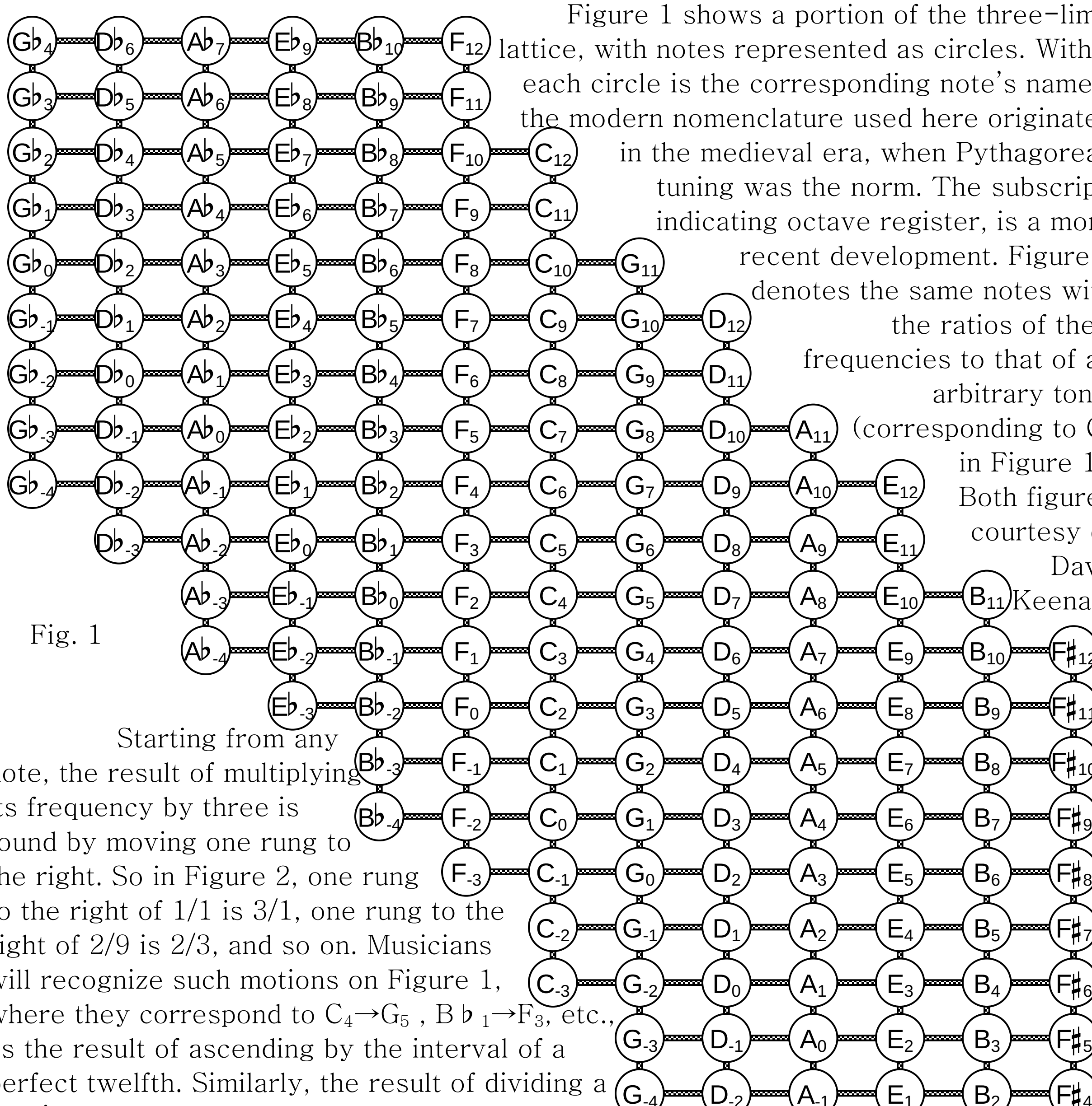
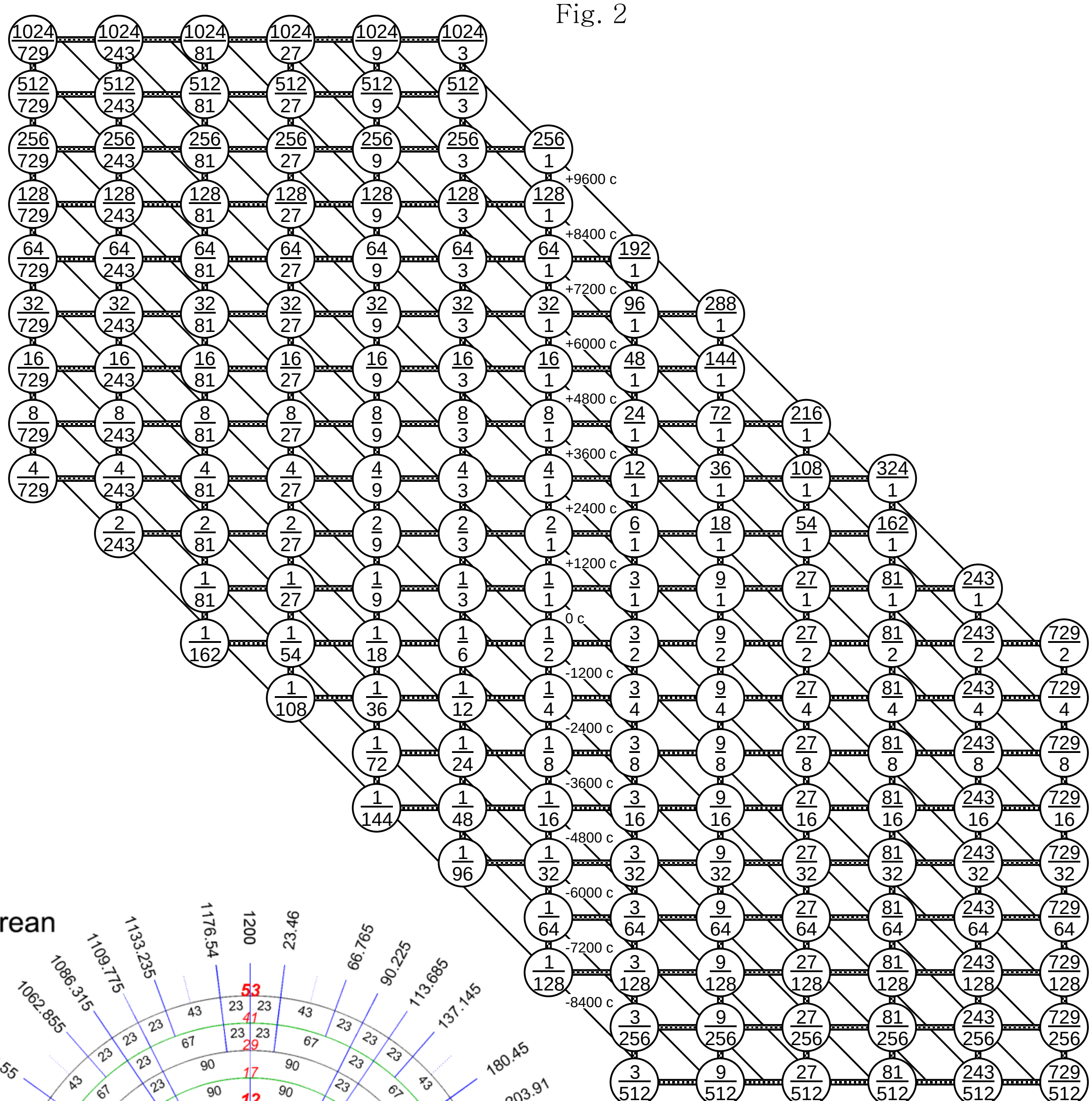


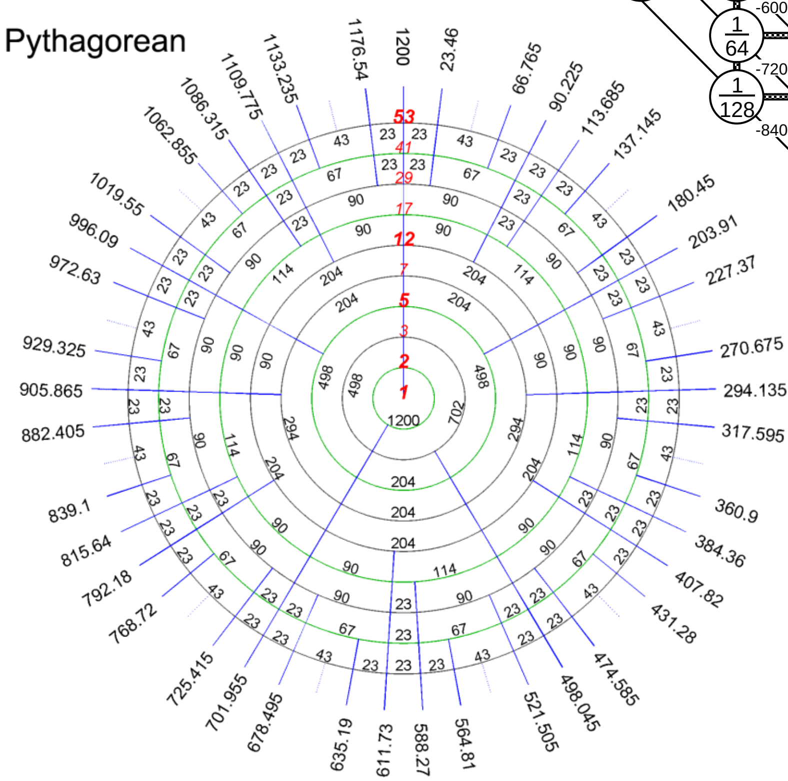
Fig. 1

Starting from any note, the result of multiplying its frequency by three is found by moving one rung to the right. So in Figure 2, one rung to the right of 1/1 is 3/1, one rung to the right of 2/9 is 2/3, and so on. Musicians will recognize such motions on Figure 1, where they correspond to $C_4 \rightarrow G_5$, $B\flat_1 \rightarrow F_3$, etc., as the result of ascending by the interval of a perfect twelfth. Similarly, the result of dividing a note's frequency by 3 is found by moving one rung

Fig. 2



1024/729 1024/243 1024/81 1024/27 1024/9 1024/3
512/729 512/243 512/81 512/27 512/9 512/3
256/729 256/243 256/81 256/27 256/9 256/3 256/1
128/729 128/243 128/81 128/27 128/9 128/3 128/1
64/729 64/243 64/81 64/27 64/9 64/3 64/1 192/1
32/729 32/243 32/81 32/27 32/9 32/3 32/1 96/1 288/1
16/729 16/243 16/81 16/27 16/9 16/3 16/1 48/1 144/1
8/729 8/243 8/81 8/27 8/9 8/3 8/1 24/1 72/1 216/1
4/729 4/243 4/81 4/27 4/9 4/3 4/1 12/1 36/1 108/1 324/1
2/243 2/81 2/27 2/9 2/3 2/1 6/1 18/1 54/1 162/1
1/81 1/27 1/9 1/3 1/1 3/1 9/1 27/1 81/1 243/1
1/162 1/54 1/18 1/6 1/2 3/2 9/2 27/2 81/2 243/2 729/2
1/108 1/36 1/12 1/4 3/4 9/4 27/4 81/4 243/4 729/4
1/72 1/24 1/8 3/8 9/8 27/8 81/8 243/8 729/8
1/144 1/48 1/16 3/16 9/16 27/16 81/16 243/16 729/16
1/96 1/32 3/32 9/32 27/32 81/32 243/32 729/32
1/64 3/64 9/64 27/64 81/64 243/64 729/64
1/128 3/128 9/128 27/128 81/128 243/128 729/128
3/256 9/256 27/256 81/256 243/256 729/256
3/512 9/512 27/512 81/512 243/512 729/512
+9600 c
+8400 c
+7200 c
+6000 c
+4800 c
+3600 c
+2400 c
+1200 c
0 c
-1200 c
-2400 c
-3600 c
-4800 c
-6000 c
-7200 c
-8400 c



Pythagorean
53
41
29
17
12
7
5
3
2
1
1200
1200
23.46
66.765
90.225
113.685
137.145
180.45
203.91
227.37
270.675
294.135
317.595
360.9
384.36
407.82
431.28
474.585
498.045
521.505
564.81
588.27
611.73
635.19
678.495
701.955
725.415
768.72
792.18
815.64
839.1
882.405
905.865
929.325
972.63
996.09
1019.55
1062.855
1086.315
1109.775
1133.235
1176.54

to the left. So one rung to the left of 1/1 is 1/3, one rung to the left of 12/1 is 4/1, etc. (Musicians will recognize such motions on Figure 1 as the result of descending by the interval of a perfect twelfth.) Hence the rightward direction will be considered the positive direction along the three-axis; the leftward direction will be considered the negative direction along the three-axis. Similarly, as one can see in Figure 2, the upward direction represents the positive direction along the two-axis; the downward direction represents the negative direction along the two-axis. Musicians, referring to Figure 1, will recognize these motions as ascents and descents, respectively, by the interval of a perfect octave.

Figure 2 also contains diagonal *pitch contours* which depict how the pitches of the notes (relative to 1/1) vary as one moves around the lattice. (The diagonal lines may appear warped but that is only an illusion.) The pitch contours show most clearly that a step in the upward direction in the lattice represents a rise of 1200 cents in pitch; a step downward, a fall of 1200 cents. The note shown as G_4 in Figure 1 and as 3/2 in Figure 2 can be seen to be a bit closer to the +1200-cent pitch contour than to the 0-cent pitch contour. If one imagines a set of pitch contours of arbitrarily fine gradation, this note would very nearly lie on the +701.955-cent contour. All the notes with a subscript of 4 in Figure 1 lie between the 0-cent and +1200-cent contours in Figure 2. Below Figure 2 is a horagram (horagrams will be explained more fully later) of Pythagorean tuning, which can be understood as showing in "rolled-up" form the pitch contours intersecting the notes falling between the 0- and +1200-cent pitch contours. The inner rings correspond to the notes within a narrow range around C_4 = 1/1; the outer rings correspond to a wider range, well beyond the limits of Figures 1 and 2.

We can identify each Pythagorean interval with a set of moves in the lattice. The interval 531441:524288, known as the Pythagorean comma, occurs for example as the interval between the pitches 1024/729 and 729/512, since (729/512):(1024/729) = 531441:524288. Observe on Figure 2 that, to get from 1024/729 to 729/512 ($G\flat_4$ to $F\#_4$ in Figure 1) in the lattice, one must move

19 rungs in the negative direction along the two-axis (downwards), and

12 rungs in the positive direction along the three-axis (to the right)

(the order of these thirty-one operations does not matter). This set of instructions for moving in the lattice is expressed in compact form by the ordered list, or *vector*,[xv]

[-19 12>.

No matter which pitch in the lattice we start at, moving according to these instructions will land you at a pitch one Pythagorean comma higher.

Arithmetically, the Pythagorean comma's ratio, 531441:524288, can be factored as

$$(3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot3)/(2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2) = 3^{12}/2^{19} = 2^{-19}\cdot3^{12}.$$

The exponents appearing over the prime numbers in the factorization of the ratio can be placed, in order, into a vector. So in vector form, the Pythagorean comma is represented again as [-19 12>. For any prime number *p*, multiplying a ratio by *p* results in adding 1 to the exponent on *p* in the ratio's factorization; dividing by *p* means subtracting 1 from that exponent. This is why the geometric and arithmetic vector representations of an interval are identical.

Every possible pitch or interval in the system can be uniquely identified with such a vector of integers. There is a one-to-one correspondence between such vectors and three-limit JI ratios. Hence three-limit JI is a 2-dimensional tuning system. In general, a JI system has one dimension for each prime number not exceeding the system's prime number limit.

The vector representation also gives us a way of computing the pitch difference, in cents, of the interval. As the pitch contours of Figure 2 illustrate, a rung in the positive direction along the two-axis represents an interval of 1200 cents, and a rung in the positive direction along the three-axis represents 1901.96 cents. Since the pitch contours are parallel and equidistant, one can simply step through the rungs needed to traverse an interval and add their pitch differences to obtain the pitch difference of that interval. Thus the Pythagorean comma is an interval of

$$-19\cdot1200 + 12\cdot1901.96 = 23.5 \text{ cents.}$$

As one can verify by looking carefully at Figure 2, moving by the Pythagorean comma takes one, in net, very slightly closer to the next-higher-octave's pitch contour – if the pitch contours were gradated finely enough and the circles drawn small enough, the 23.5-cent difference could be visually confirmed. A convenient shorthand for writing this expression is

<1200 1901.96|-19 12> = 23.5;

this is an example of a *bracket product* operation. The object on the left of the expression, which can be written in isolation as

<1200 1901.96]

is sometimes referred to as a *covector*; covectors are a convenient way of representing structures like the set of pitch contours in Figure 2.[xvi]

The Five-Limit JI Lattice

We now add an additional prime – five – and thus an additional dimension to our JI universe. Below are two representations of a portion of the five-limit lattice, with notes represented as "balls". Figure 3 gives the conventional names of these pitches (with Sagittal[xvii] symbols denoting syntonic-comma inflections and subscripted numerals, as before, denoting octave register). Figure 4 denotes the same pitches with the ratios of their frequencies to that of an arbitrary tonic (corresponding to C_4 in Figure 3). Both figures were contributed by Dave Keenan.

Just as in Figures 1 and 2, the horizontal direction in these lattices represents the three-axis. Starting from any note, the result of multiplying its frequency by three (i.e., rising by a perfect twelfth) is found by moving one rung to the right. So one rung to the right of 1/1 is 3/1, one rung to the right of 2/15 is 2/5, and so on. Similarly, the result of dividing a note's frequency by 3 (i.e., of descending a perfect twelfth from that note) is found by moving one rung to the left. So one rung to the left of 1/1 is 1/3, one rung to the left of 12/5 is 4/5, etc. Hence the rightward direction will be again considered the positive direction along the three-axis; the leftward direction will be considered the negative direction along the three-axis. Similarly, as one can see in Figure 4, the upward direction represents the positive direction along the five-axis; the downward direction represents the negative direction along the five-axis.

Fig. 3

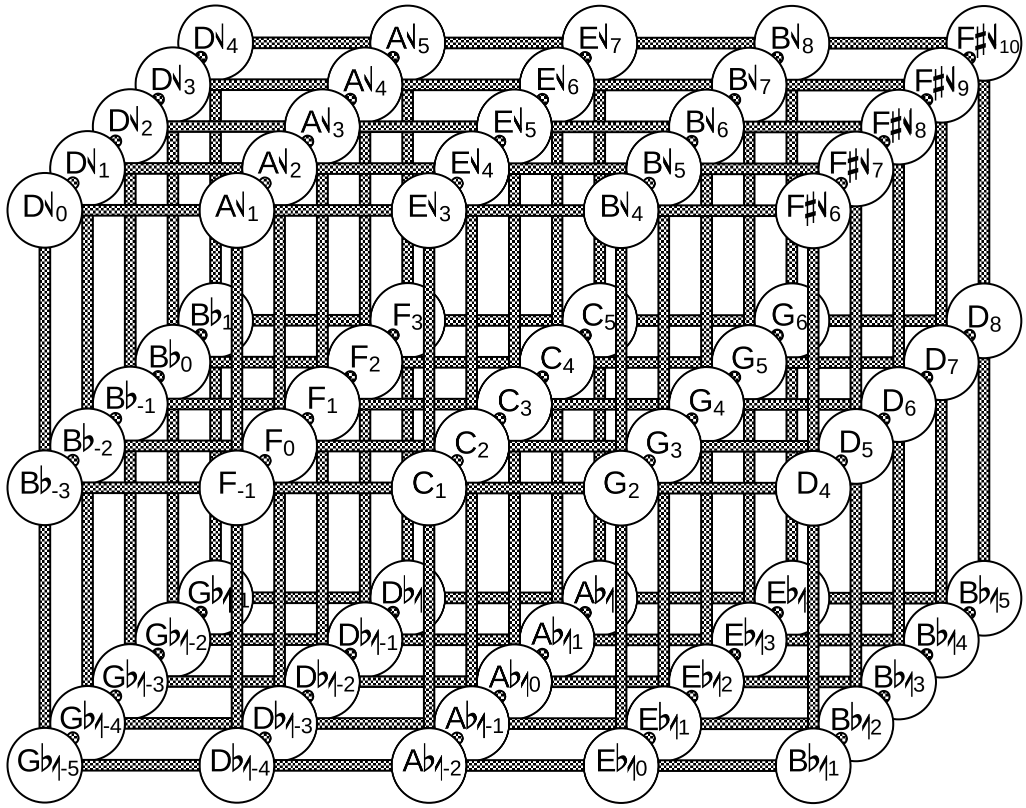

B♭-3
F-1
C1
G2
D4
B♭-2
F0
C2
G3
D5
B♭-1
F1
C3
G4
D6
B♭0
F2
C4
G5
D7
B♭1
F3
C5
G6
D8

Fig. 4

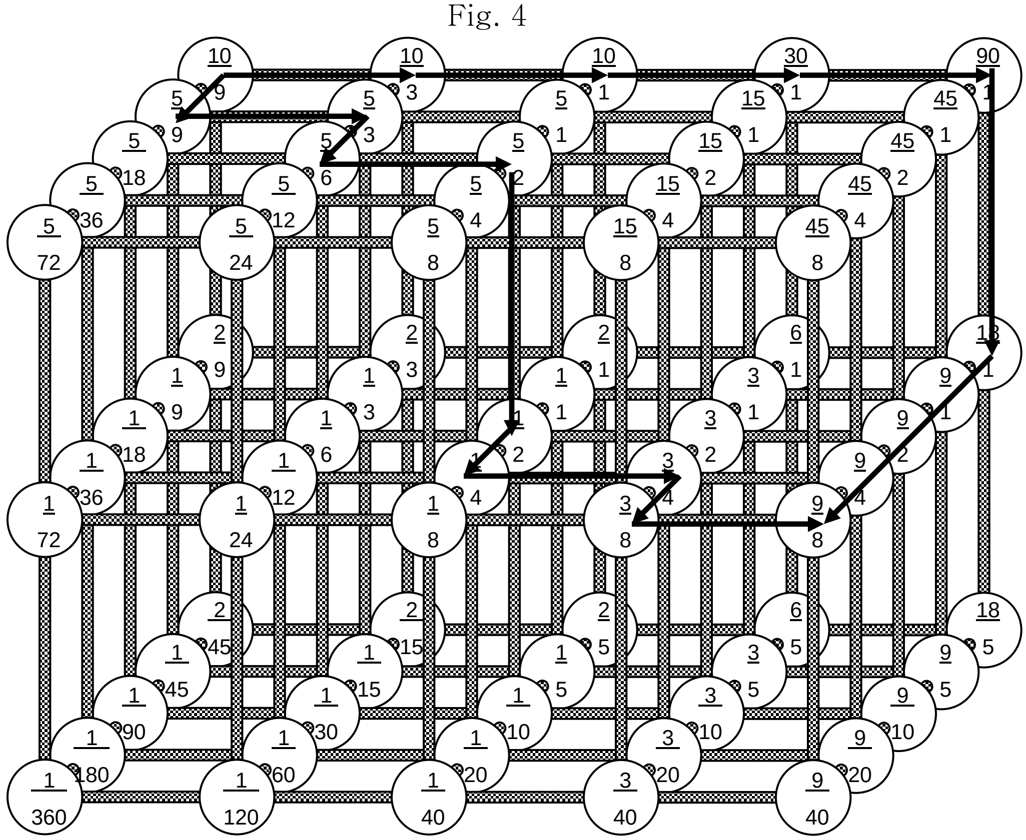

10/9
10/3
10/1
30/1
90/1
5/9
5/3
5/1
15/1
45/1
5/18
5/6
5/2
15/2
45/2
5/36
5/12
5/4
15/4
45/4
5/72
5/24
5/8
15/8
45/8
2/9
2/3
2/1
6/1
18/1
1/9
1/3
1/1
3/1
9/1
1/18
1/6
1/2
3/2
9/2
1/36
1/12
1/4
3/4
9/4
1/72
1/24
1/8
3/8
9/8
2/45
2/15
2/5
6/5
18/5
1/45
1/15
1/5
3/5
9/5
1/90
1/30
1/10
3/10
9/10
1/180
1/60
1/20
3/20
9/20
1/360
1/120
1/40
3/40
9/40

Figure 3 shows that the vertically oriented rungs represent the musical interval of a major seventeenth, narrowed relative to its Pythagorean tuning by a syntonic comma. Finally, it should be clear that the direction that appears to go "into the page" represents the positive direction along the two-axis, and the direction that appears to come "out of the page" represents the negative direction along the two-axis – each rung representing the interval of a perfect octave.

As in the Pythagorean case, we can identify each JI interval with a set of moves in the lattice. The syntonic comma, 81:80, occurs for example as the interval between the notes 10/9 and 9/8, since (9/8):(10/9) = 81:80. Two paths from 10/9 to 9/8 are shown in Figure 4. To get from 10/9 to 9/8 in the lattice, one must move

4 rungs in the negative direction along the two-axis ("out of the page"),
4 rungs in the positive direction along the three-axis (to the right), and
1 rung in the negative direction along the five-axis (down)

(the order of these nine operations does not matter). This set of instructions for moving in the lattice is expressed in compact form by the vector [-4 4 -1>. No matter which note in the lattice we start at, moving according to these instructions will land you at a pitch one syntonic comma higher.

Arithmetically, the syntonic comma's ratio, 81/80, can be factored as

$$(3\cdot3\cdot3\cdot3)/(2\cdot2\cdot2\cdot2\cdot5) = 3^4/(2^4\cdot5^1) = 2^{-4}\cdot3^4\cdot5^{-1}.$$

So in vector form, the syntonic comma is represented again as [-4 4 -1>. To review: for any prime number *p*, multiplying a ratio by *p* results in adding 1 to the exponent on *p* in the ratio's prime-factorization; dividing by *p* means subtracting 1 from that exponent – this is why the geometric and arithmetic vector representations of an interval are identical. And as before, every possible pitch or interval in the system can be uniquely identified with such a vector of integers. There is a one-to-one correspondence between such vectors and five-limit JI ratios. Hence five-limit JI is a 3-dimensional tuning system.[xviii]

The vector representation, as before, gives us a way of computing the pitch difference, in cents, of the interval. In JI, a rung in the positive direction along the two-axis represents an interval of 1200 cents. A rung in the positive direction along the three-axis represents 1901.96 cents. And a rung in the positive direction along the five-axis represents an interval of 2786.31 cents. Thus the syntonic comma is an interval of

$$-4\cdot1200 + 4\cdot1901.96 + -1\cdot2786.31 = 21.5 \text{ cents.}$$

Using the bracket product shorthand, this expression can be written as

$$\langle 1200\ 1901.96\ 2786.31 | -4\ 4\ -1 \rangle = 21.5.$$

Though they are not shown in the figures above, the five-limit lattice should be thought of as equipped with a set of pitch contours just as the three-limit lattice can. They form a set of parallel, equidistant planes that cut diagonally through the lattice, and can be denoted by the covector

$$\langle 1200\ 1901.96\ 2786.31].$$

The Tenney Lattice and Harmonic Distance[xix]

In our considerations so far, the rungs of the just intonation lattice, corresponding to the prime numbers, were not presumed to have any particular lengths relative to one another. If one imagines the rungs as each having length proportional to the corresponding prime number's interval-size (and figures 1-4 are drawn so that this looks plausible), one has the *Tenney lattice*.

As we saw at the end of each of the last two sections, the JI lattice provides a convenient way to measure the difference in pitch between any two notes in the system. Express the interval between the two notes as a vector of integers, representing the signed number of rungs along each prime number's axis required to traverse the interval. Then multiply each component of this vector by the corresponding component in the covector of the primes' interval-sizes (in, say, cents), and add the products together. If we use the absolute values[xx] of the number of rungs along each axis instead of the signed numbers, again multiplying component-by-component with the corresponding primes' interval-sizes (in, say, cents), we are measuring the *taxicab* or *Manhattan* distance of the interval in the Tenney lattice. This distance function is so called because, like a taxicab traversing the streets of Manhattan, one moves through the lattice only along the edges, and never takes a "shortcut" that cuts diagonally through a block. For example, if the rungs' lengths are taken to be equal to the corresponding intervals in cents, the syntonic comma has a harmonic distance of

$$4\cdot 1200 + 4\cdot 1901.96 + 1\cdot 2786.31 = 15194.1$$

To prevent the numbers from getting unwieldy, the length of one rung along the two-axis can be taken to be 1 in this calculation, so we divide each term above by 1200 – or equivalently, the primes' interval-sizes could have been measured in octaves instead of cents. The result is

$$15194.1/1200 = 12.662 = \log_2(81\cdot 80)$$

for the *harmonic distance* (HD) of the syntonic comma 81/80.

Harmonic distance is not a measure of interval size. Instead of measuring the given interval in cents or octaves – a quantity proportional to log(n/d), where n is the interval-ratio's numerator and d its denominator – one obtains a quantity proportional to log(n·d). In 1563, Benedetti proposed that n·d provides a ranking of the simple ratios from most consonant to most dissonant, for example giving the ordering 2/1, 3/2, 4/3, 5/3, 5/4, 6/5, 8/5, 9/5. Log(n·d) gives the same ranking, but agrees with harmonic entropy[xxi] as to the proportions of the differences in concordance among these intervals. Beyond the simple ratios, such simple numerical rankings of interval concordance break down, but the quantity remains useful as a measure of the musical complexity of a JI harmonic progression required to traverse the interval. This is why, in Tenney's terminology, the taxicab distance an interval traverses in his lattice is the "Harmonic Distance" of that interval. While expressing intervals as vectors instead of ratios can be very useful, the size of the numbers in the intervals' ratios gives a direct indication of the intervals' harmonic distance.

Temperament

Temperament[xxii] then consists in altering the tuning of the prime-number intervals so that some of the intervals in the tuning system become perfect unisons (pitch difference of zero). One can visualize this as a reorientation of the pitch contours so that, if a pitch contour passes through one note in the lattice, it will also pass through others (which is not the case in just intonation). For example, in investigating Pythagorean tuning, we found that 729/512 lies on a slightly higher pitch contour than 1024/729. Now let us imagine that the pitch contours angle slightly differently, so that these two notes now lie on the same pitch contour. This must mean that if we move by the interval between these two notes, the Pythagorean comma, we will now end up on the same pitch contour where we began. In other words, since the Pythagorean comma is represented by the vector [-19 12>, moving 19 rungs in the negative direction along the two-axis and 12 rungs in the positive direction along the three-axis will now return us to the same pitch we started on. Thus temperament introduces some redundancy into the lattice representation of pitches. This so-called "vanishing" of the Pythagorean comma thus implies that moving 19 rungs in the positive direction along the two-axis is equivalent to moving 12 rungs in the positive direction along the three-axis; so moving one rung along the three axis is equivalent to moving 19/12 of a rung along the two-axis. Since adding or subtracting 19/12 to an integer results in a multiple of 1/12, it becomes possible to represent all intervals in the tuning as multiples of 1/12 of a rung along the two-axis.

Evidently, we have come upon a system where only one integer is needed to specify any interval in the tuning – a 1-dimensional tuning, where all intervals are multiples of 1/12 of prime two – namely 12-tone equal temperament. Indeed, the early Chinese discovery of 12-equal was predicated on an ideal of three-limit JI. By tempering out (or causing to vanish) one interval from three-limit JI, we've reduced the dimensionality of the system from 2 to 1. We can therefore depict the tuning system along a single line as follows:

$$\cdots E\flat_3 - E_3 - F_3 - G\flat_3 - G_3 - A\flat_3 - A_3 - B\flat_3 - B_3 - C_4 - D\flat_4 - D_4 - E\flat_4 - E_4 - F_4 - G\flat_4 - G_4 - A\flat_4 - A_4 \cdots$$

(where of course G♭ is now synonymous with F#, etc.). Though this depiction fails to suggest harmonic proximity the way the Tenney lattice does, it correctly gets across the intervallic structure and dimensionality of 12-equal.

Another familiar example is meantone tuning: we begin with the 3-dimensional lattice of five-limit just intonation and adjust the tuning of prime three, (sometimes also prime five, and rarely prime two as well), in such a way that the syntonic comma vanishes. Keep in mind that this can be visualized as a reorientation of the planar pitch contours cutting through the lattice. After this adjustment, if we move

4 rungs in the negative direction along the two-axis,
4 rungs in the positive direction along the three-axis, and
1 rung in the negative direction along the five-axis,

we end up at the same pitch we started at. Algebraically, we can eliminate this redundancy by noting that moving

1 rung in the positive direction along the five-axis

is now equivalent to moving

4 rungs in the negative direction along the two-axis and
4 rungs in the positive direction along the three-axis

(in whatever order). Hence we can replace all motions along the five-axis with equivalent motions along the other 2 axes: the two-axis and the three-axis. In this case these two motions will each involve an integer number of rungs. We thus end up with a 2-dimensional system. Only two integers are necessary to specify any interval in the tuning: an integer giving the number (and direction) of steps along the two-axis, and an integer giving the number (and direction) of steps along the three-axis. Fortuitously, we already have exactly such a depiction of a tuning system. Figure 1 can now be re-interpreted as a depiction of meantone tuning, though in this guise it now fails to give an accurate suggestion of harmonic proximity as it did in the Pythagorean case. It does, however, correctly get across the intervallic structure and dimensionality of meantone tuning. Had we tempered out a different interval, rather than the syntonic comma, from 5-limit JI, we might not have ended up with a system so similar to Pythagorean tuning, but a 2-dimensional system would still have been the result.

In general, for each independent interval that vanishes, the dimensionality of the tuning system decreases by one. Western music had been based on an assumption of meantone temperament since the late 15th century. By the turn of the 19th, the influence of Bach, Beethoven and others had precipitated a further decrease in dimensionality, from two to one. Since in meantone temperament the syntonic comma already vanished, one choice of an additional JI interval to temper out would be just as good as another if they merely differed by one or more syntonic commas. The relevant possibilities for the Western example – intervals separating enharmonic pairs of pitches such as F# and G♭ – include the dieses 648:625 and 128:125, the diaschisma 2048:2025, the schisma 32805:32768, and the Pythagorean comma 531441:524288.[xxiii] As we saw above, it is possible to specify all of meantone tuning with only 2 integers – rungs along the two-axis and rungs along the three-axis. Therefore, the vanishing of the Pythagorean comma now acts essentially as it did in the three-limit case, resulting in the 1-dimensional system of 12-tone equal temperament.

Under the usual restriction that the two-axis remains just and the other axis or axes take all the tempering, these two 12-tone equal tempered systems – the Chinese one derived from three-limit JI, and the Western one derived from five-limit JI – are tuned identically. But this is not necessarily the only reasonable alternative. One can tailor the tempering so as to do, in a sense, the least possible damage to the harmonies in the original JI lattice. We will discuss such a procedure in the next section.

TOP Tuning

During a recent bout of illness which provided much time to think, it seemed to me[xxiv] that if only one (independent) interval vanishes, the most natural tuning is the one where the vanishing interval's taxicab route undergoes the tempering *uniformly* along its length. That is, one adjusts the tunings of the rungs involved in traversing the interval a fixed amount per unit length, but (assuming the interval is a positive pitch difference in JI) with sign *opposite* to the direction in which they are traversed (so that the pitch difference ends up being zero).

Let's make this precise for the meantone example. Recall that the syntonic comma traverses 4 rungs in the negative direction along the two-axis, 4 rungs in the positive direction along the three-axis, and 1 rung in the negative direction along the five-axis. In order to make the syntonic comma vanish without "harmonic waste", we should therefore lengthen the negatives and shorten the positives. The tempering each rung undergoes, according to the idea above, will equal the tempering of the comma (its entire pitch difference, since it's being set to zero) times the length of the rung as a fraction of the taxicab length of the comma. So the two-axis rungs should be tempered wide by

$$21.5 \text{ cents} \cdot \log(2)/\log(81 \cdot 80) = 1.7 \text{ cents},$$

making them 1201.7 cents; the three-axis rungs should be tempered narrow by

$$21.5 \text{ cents} \cdot \log(3)/\log(81 \cdot 80) = 2.7 \text{ cents},$$

making them 1899.26 cents; and the five-axis rungs should be tempered wide by

$$21.5 \text{ cents} \cdot \log(5)/\log(81 \cdot 80) = 3.94 \text{ cents},$$

making them 2790.26 cents. Verifying that the syntonic comma vanishes, we repeat our earlier bracket product calculation with the covector corresponding to the new rungs

$$\langle 1201.7\ 1899.26\ 2790.26 | -4\ 4\ -1 \rangle = 0.$$ [xxv]

I then realized that this method of tuning was *optimal* in a certain sense. Most models of discordance predict that the simplest ratios are most sensitive to mistuning, more complex ratios are less sensitive to mistuning, and still more complex ratios are essentially insensitive to mistuning (as they are not local minima of discordance in the first place). To evaluate the damage to concordance caused by the mistunings in a temperament, then, it makes sense to scale them so that mistuning a simple ratio by a given amount (in cents) corresponds to more damage than mistuning a complex ratio by that same amount. A straightforward way of doing this is to divide each mistuning by the Harmonic Distance of the JI interval mistuned. Clearly we have tempered out the comma so that the maximum damage done to the three prime-number intervals is minimized. Any other way of distributing the comma among the primes, if it decreased the damage on one of them, would have to increase the damage to another, thus increasing the maximum.

However, the primes are not the only intervals of interest. Let's examine the meantone tuning derived above in terms of the damage it does to the five-prime-limit ratios with Harmonic Distance less than 6:

IntervalRatio	Tmprmnt.	$\lvert$Mistuning$\rvert$	HD=$\log_2(n*d)$	$\lvert$Mistuning$\rvert$/HD
2/1	**1201.70**	**1.70**	**1**	**1.70**
3/1	1899.26	2.69	1.58	1.70
4/1	2403.40	3.40	2	1.70
5/1	2790.26	3.94	2.32	1.70
3/2	**697.56**	**4.39**	**2.58**	**1.70**
6/1	3100.96	0.99	2.58	0.38
8/1	3605.10	5.10	3	1.70
9/1	3798.53	5.38	3.17	1.70
5/1	2790.26	3.94	2.32	1.70
10/1	3991.96	5.64	3.32	1.70
4/3	**504.13**	**6.09**	**3.58**	**1.70**
12/1	4302.66	0.70	3.58	0.20
5/3	**890.99**	**6.64**	**3.91**	**1.70**
15/1	4689.52	1.25	3.91	0.32
16/1	4806.79	6.79	4	1.70
9/2	2596.83	7.08	4.17	1.70

18/1	5000.22	3.69	4.17	0.88
5/4	**386.86**	**0.55**	**4.32**	**0.13**
20/1	5193.65	7.34	4.32	1.70
8/3	1705.83	7.79	4.58	1.70
24/1	5504.36	2.40	4.58	0.52
25/1	5580.52	7.89	4.64	1.70
27/1	5697.79	8.08	4.75	1.70
6/5	**310.70**	**4.94**	**4.91**	**1.01**
10/3	2092.69	8.33	4.91	1.70
15/2	3487.82	0.45	4.91	0.09
30/1	5891.22	2.95	4.91	0.60
32/1	6008.49	8.49	5	1.70
9/4	1395.13	8.78	5.17	1.70
36/1	6201.92	1.99	5.17	0.38
8/5	**814.84**	**1.15**	**5.32**	**0.22**
40/1	6395.35	9.04	5.32	1.70
9/5	**1008.27**	**9.33**	**5.49**	**1.70**
45/1	6588.78	1.44	5.49	0.26
16/3	2907.53	9.49	5.58	1.70
48/1	6706.06	4.10	5.58	0.73
25/2	4378.82	6.19	5.64	1.10
50/1	6782.21	9.59	5.64	1.70
27/2	4496.09	9.77	5.75	1.70
54/1	6899.49	6.38	5.75	1.11
12/5	1512.40	3.24	5.91	0.55
15/4	2286.12	2.15	5.91	0.36
20/3	3294.39	10.03	5.91	1.70
60/1	7092.92	4.65	5.91	0.79

The largest amount of damage done to any interval here is 1.7 by this measure, the same as the damage to the prime-number intervals. This would remain true no matter how complex the interval ratios we considered susceptible to 'damage'.[xxvi] Our tuning strategy has minimized the maximum damage over all intervals in the lattice. This is a very convenient property to have in a temperament optimization as it requires only the specification of a prime limit and not a more restrictive odd limit, integer limit, or other putative set of 'concordant' intervals.[xxvii]. The tuning would also remain optimal if we restricted our attention to intervals no larger than an octave (shown in boldface above), for example.

Thus the general rule for Tenney-optimal temperament of a single comma is as follows. Make sure the comma, n/d is a positive pitch difference in JI. i.e., n is greater than d. Next, consider each of the prime numbers in succession. If the prime p has a positive sign in the comma's vector representation – that is, if it appears as a factor of n – then temper that prime narrow by

$$\text{cents}(n/d) \cdot \log(p)/\log(n \cdot d).$$

If the prime p has a negative sign in the comma's vector representation – that is, if it appears as a factor of the d – then temper that prime wide by this same amount. If the comma does not involve prime p at all, we will adopt the convention that prime p remains just.[xxviii] The maximum damage of any interval, which is minimized by this temperament strategy, is then

$$\text{cents}(n/d) \cdot \log(2)/\log(n \cdot d)$$

or

$$1200 \cdot \log(n/d)/\log(n \cdot d).$$

We can apply this strategy to the Chinese 12-equal example because only one comma, the Pythagorean comma, is tempered out there. The Pythagorean comma traverses 19 rungs in the negative direction along the two-axis and 12 rungs in the positive direction along the three-axis. It is a pitch difference in JI of

$$\langle 1200\ 1901.96 | -19\ 12 \rangle = 23.5 \text{ cents.}$$

So we temper the two-axis rungs wide by

$$23.5 \text{ cents} \cdot \log(2)/\log(531441 \cdot 524288) = 0.62 \text{ cents,}$$

making them 1200.62 cents, and the three-axis rungs narrow by

$$23.5 \text{ cents} \cdot \log(3)/\log(531441 \cdot 524288) = 0.98 \text{ cents}$$

making them 1900.98 cents. Observe that the Pythagorean comma vanishes:

$$\langle 1200.62\ 1900.98 | -19\ 12 \rangle = 0.$$[xxix]

We can also verify that this is an equal tuning as expected:

$$1200.62/12 = 100.05 = 1900.98/19.$$

This idea can be generalized to a method of optimal tuning for any class of temperament (no matter how many commas vanish). This is now known as TOP tuning – "Tenney OPtimal".[xxx] Another interpretation of this acronym is "Tempered Octaves, Please", as almost all the types of optimal tuning my colleagues and I had considered until this year had pure octaves. The precise tuning details, though, should not be considered essential – all the temperaments in this paper can be tuned in various ways with pure octaves, irregular or "well-" tempering, or other with features that deviate from the TOP model, according to the needs and desires of the musician.

Linear Temperaments, Moment-Of-Symmetry scales, and Horagrams?

Earlier, we found that any pitch or interval in meantone tuning can be uniquely specified using two integers, representing numbers of rungs along the two-axis and three-axis, respectively. So we will say that the meantone system can be *generated* by its approximations of the 2:1 and 3:1 ratios, or that these approximate ratios form one possible pair of *generators* of meantone tuning. Tuning systems are usually constructed so as to repeat themselves at intervals of ~ (approximately) 2:1 (the "octave"). So the ~2:1 generator of meantone functions as the *period* of the system.

In musical cultures whose scales repeat periodically at the ~2:1, such as the West, pitches separated by ~2:1 are treated as "equivalent", typically having the same name. The study of tuning systems is often undertaken with equivalence classes of pitches, rather than pitches themselves, as the fundamental entities. So meantone's ~3:1 generator, or equivalently 3:2 (the "fifth") or 3:4 (the "fourth"), can often be regarded simply as *the* generator of meantone. In this guise, meantone temperament, though 2-dimensional, has been referred to as a "linear temperament" by Erv Wilson and others.

It's not always the case, though, that a 2-dimensional temperament can be generated by a pair of intervals such that one of them is ~2:1. We saw that 12-tone equal temperament requires a generator that is one-twelfth of an octave, and this sort of thing can happen with two-dimensional temperaments as well. Specifically, if a 'comma' that vanishes in the temperament can be expressed as the difference between some number of "octaves" and a stack of identical JI intervals, the temperament will contain fraction-of-octave intervals that cannot be expressed using a whole octave as one of the generators. For example, the diaschisma, 2048:2025 (in vector notation, [11 -4 -2>), can be expressed as the difference between an octave, 2/1 ([1 0 0>) and a stack of two identical "just augmented fourths", each 45:32 ([-5 2 1>). If the diaschisma vanishes, two "augmented fourths" will become equal to one "octave". The "octave" cannot, therefore, itself be a generator of the resulting temperament, but a "half-octave" can. With octave-repetition assumed, the half-octave becomes the period. The dieses 128:125 ([7 0 -3>) and 648:625 ([3 4 -4>), when tempered out, can similarly be seen to result in "third-octave" and "quarter-octave" periods, respectively. Such fractions-of-octave periods, though, are clearly not intervals of equivalence, either psychoacoustically or culturally. So the applicability of the term "linear temperament" to all 2-dimensional temperaments is dubious.

Once the period and generator of a 2-dimensional temperament are identified, putative scales can be created in a straightforward way. Starting with one note per period, the generator is used to add more and more pitches to the scale. At certain points in this process, the scale has only two different step sizes. At these points, traversing any given number of consecutive steps in the scale can result in at most two specific sizes of interval, regardless of where one begins in the scale. The number of sizes will be exactly two except where the interval is the period or a multiple thereof (in which case there's only one size). In cases where the period is an octave, these points in the process have been referred to as Moments-Of-Symmetry, or MOS scales, by Erv Wilson and others. They have also been called Myhill scales. A more general term, communicated to me by the late John Clough, that includes the cases where the octave is a multiple number of periods is Distributionally Even Scale (DES).

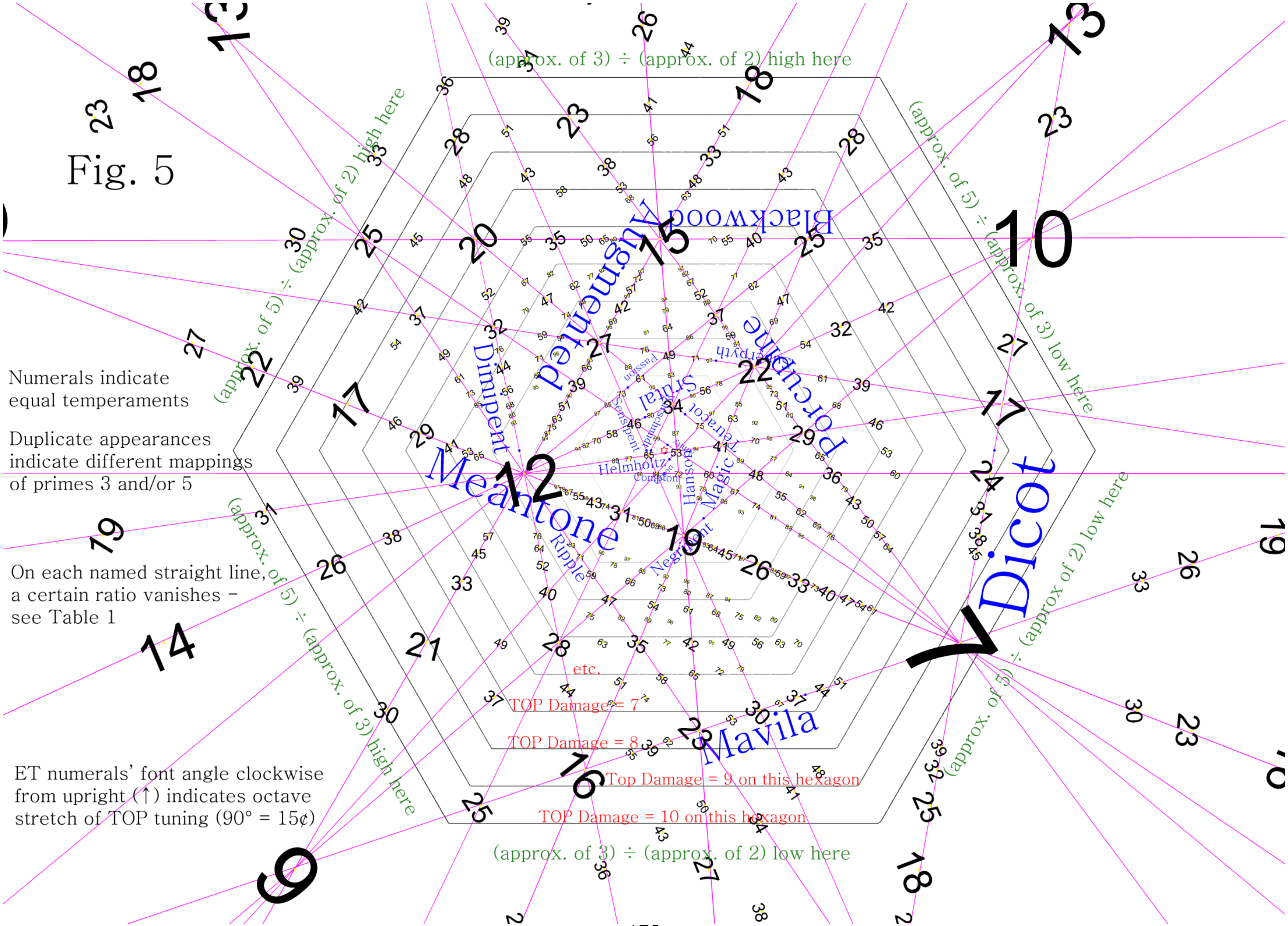
Pitches in scales that repeat every (tempered) octave can be grouped together into "equivalence classes". We'll use the term "pitch class" to indicate a pitch along with all its (tempered) octave-transpositions. Using this idea, we can concisely display DESs by employing a type of diagram called a "horagram" by Erv Wilson. He uses them to depict MOS scales but I will take the liberty of applying the term to this more general case. Take a look now at the horagrams which comprise the second half of this paper. Let's use the TOP meantone horagram as an example. The horagram depicts pitch classes as rays, and intervals as angles, with a (tempered) octave represented, like an hour on a clock, as a full circle. The diagram begins with one pitch class per period shown as a ray or rays emanating from the center of the diagram. In the TOP meantone case, this is a single vertical ray, representing 0 cents, the tempered octave of 1201.7 cents (explicitly indicated), and all integer multiples thereof. Then the process of repeatedly applying the generator to add more pitch classes occurs. In this case, the generator is the tempered fourth of 504.13 cents.

Unlike Wilson, in whose horagrams the generator is always applied in the same direction, I apply the generator alternately upward from one end of the chain(s) and downward from the other. The direction of the first generator is arbitrary – I happened to apply it downward in this case. So the second pitch generated within the frame of an octave is obtained by applying the generator downward from the first pitch class: 1201.7 - 504.13 = 697.57 cents. The third pitch obtained within the octave is obtained by applying the generator upward from the first pitch class: 0 + 504.13 = 504.13 cents. The fourth pitch class is obtained by applying the generator downward from the second pitch class: 697.57 - 504.13 = 193.44 cents. The fifth pitch class is obtained by applying the generator upward from the third pitch class: 504.13 + 504.13 = 1008.26 cents. Each time a DES is formed, a concentric ring is drawn, the number of notes per octave in the DES is printed, in italics, in the ring near the top,[xxxi] and each step is labeled with its size in cents. The process continues with the rays emanating from the outside of the ring. Successive rings get drawn further and further from the center.

TOP meantone shows 5- and 7-note rings, corresponding to the familiar pentatonic and diatonic scales. If the 0/1201.7-cent pitch class is assigned to the note D, the rest of the 7-note ring will correspond to the notes E, F, G, A, B, and C. A chain of 7 generators produces the small interval of 76.19 cents, which first shows itself in the 12-note ("chromatic") ring. This interval is also the difference between the two step sizes in the 7-note ring. Raising a note by this interval appends a "sharp" (#) to its name, while a like lowering appends a "flat" (♭). With these symbols, any meantone pitch class can be assigned a unique name.

Table 1

Vanishing Interval's Ratio	Van. Intvl. Vector	V. I. cents	Horagram name	TOP per.	TOP gen.	Map 2	Mapp. of 3	Mapp. of 5	Cmplx	TOP Dmg.
TWO EXOTEMPERAMENTS:										
15:16	[4 -1 -1>	111.7	Father	1185.9	447.4	1,0	2,-1	2,1	2.15	14.13
27:25	[0 3 -2>	133.2	Bug	1200.0	260.3	1,0	2,-2	3,-3	2.55	14.18
MAIN SEQUENCE:										
25:24	[-3 -1 2>	70.7	Dicot	1207.66	353.22	1,0	1,2	2,1	2.51	7.66
81:80	[-4 4 -1>	21.5	Meantone	1201.70	504.13	1,0	2,-1	4,-4	3.44	1.70
128:125	[7 0 -3>	41.1	Augmented	399.02	93.15	3,0	5,-1	7,0	3.79	2.94
135:128	[-7 3 1>	92.2	Mavila	1206.55	685.03	1,0	1,1	4,-3	3.83	6.55
250:243	[1 -5 3>	49.2	Porcupine	1196.91	1034.59	1,0	-1,3	-2,5	4.32	3.09
256:243	[8 -5 0>	90.2	Blackwood	238.87	158.78	5,0	8,0	11,1	4.33	5.67
648:625	[3 4 -4>	62.6	Dimipent	299.16	197.49	4,0	7,-1	10,-1	5.06	3.36
2048:2025	[11 -4 -2>	19.6	Srutal	599.56	494.86	2,0	4,-1	3,2	5.97	0.89
3125:3072	[-10 -1 5>	29.6	Magic	1201.28	380.80	1,0	0,5	2,1	6.30	1.28
6561:6250	[-1 8 -5>	84.1	Ripple	1203.32	101.99	1,0	2,-5	3,-8	6.87	3.32
15625:15552	[-6 -5 6>	8.1	Hanson	1200.29	317.07	1,0	0,6	1,5	7.57	0.29
16875:16384	[-14 3 4>	51.1	Negripent	1201.82	1075.68	1,0	-2,4	5,-3	7.62	1.82
20000:19683	[5 -9 4>	27.7	Tetracot	1199.03	176.11	1,0	1,4	1,9	7.76	0.97
20480:19683	[12 -9 1>	68.7	Superpyth	1197.60	708.17	1,0	1,1	-3,9	7.77	2.40
32805:32768	[-15 8 1>	2.0	Helmholtz	1200.07	701.79	1,0	1,1	7,-8	8.15	0.07
78732:78125	[2 9 -7>	13.4	Sensipent	1199.59	756.60	1,0	6,-7	8,-9	8.84	0.41
262144:253125	[18 -4 -5>	60.6	Passion	1198.31	98.40	1,0	2,-5	2,4	9.77	1.69
393216:390625	[17 1 -8>	11.4	Würschmidt	1199.69	812.05	1,0	7,-8	3,-1	10.10	0.31
531441:524288	[-19 12 0>	23.4	Compton	100.05	15.13	12,0	19,0	28,-1	10.33	0.62
1600000:1594323	[9 -13 5>	6.2	Amity	1199.85	860.38	1,0	-2,5	-7,13	11.20	0.15
2109375:2097152	[-21 3 7>	10.1	Orson	1200.24	271.65	1,0	0,7	3,-3	11.41	0.24
TWO BONUS TEMPS.:										
6115295232:6103515625	[23 6 -14>	4.2	Vishnu	599.97	71.15	2,0	4,-7	5,-3	17.67	0.05
274877906944:274658203125	[38 -2 -15>	1.4	Luna	1199.98	193.196	1,0	4,-15	2,2	20.65	0.02


Numerals indicate equal temperaments
Duplicate appearances indicate different mappings of primes 3 and/or 5
On each named straight line, a certain ratio vanishes – see Table 1
ET numerals' font angle clockwise from upright (↑) indicates octave stretch of TOP tuning (90° = 15¢)
(approx. of 3) ÷ (approx. of 2) high here
(approx. of 3) ÷ (approx. of 2) low here
(approx. of 5) ÷ (approx. of 2) high here
(approx. of 5) ÷ (approx. of 2) low here
(approx. of 5) ÷ (approx. of 3) high here
(approx. of 5) ÷ (approx. of 3) low here
TOP Damage = 7
TOP Damage = 8
Top Damage = 9 on this hexagon
TOP Damage = 10 on this hexagon
etc.
Meantone
Dicot
Mavila
Augmented
Blackwood
Porcupine
Dimipent
Srutal
Magic
Negri
Ripple
Hanson
Tetracot
Helmholtz
Compton
Sensipent
Schismic
Passion
Superpyth
Fig. 5

A similar notation scheme can be devised for any 2-dimensional temperament. The reader may find it useful or enjoyable to concoct such schemes using the horagrams here provided.

Table 1 provides data on all the 5-limit 2-dimensional temperaments for which horagrams are provided. The first column gives the ratio of the interval whose vanishing defines the temperament.[xxxii] The second column gives this interval in vector form, and the third column gives the interval's untempered (JI) size in cents. The fourth column is the name used to label the horagram. The fifth and sixth columns give the period and generator, respectively. The next three columns show how the primes 2, 3, and 5 are approximated by combining an integer number of periods and an integer number of generators. For example, prime 5 in meantone is approximated by ascending 4 periods and descending 4 generators, so the "mapping of 5" is given as "4, -4". The second-to-last column gives the complexity of the temperament, a measure of the size of the portion of the five-limit Tenney lattice that is required to represent the entire temperament (the whole lattice isn't needed since temperament introduces redundancy).[xxxiii] The last column gives the maximum damage incurred by the TOP tuning.

Table 2

Vanishing Intervals' Ratios	Horagram name	TOP per.	TOP gen.	Map of 2	Mapp. of 3	Mapp. of 5	Mapp. of 7	Cmplx	TOP Dmg
28:27, 49:48, 64:63, 256:243, 343:324,...	Blacksmith	239.18	155.35	5,0	8,0	11,1	14,0	6.47	7.24
36:35, 50:49, 126:125, 360:343, 648:625,..	Dimisept	298.53	197.08	4,0	7,−1	10,−1	12,−1	7.92	5.87
36:35, 64:63, 81:80, 256:245, 729:700,...	Dominant	1195.23	495.88	1,0	2,−1	4,−4	2,2	7.96	4.77
36:35, 128:125, 225:224, 405:392, 729:686,...	August	399.99	107.31	3,0	5,−1	7,0	9,−2	8.30	5.87
50:49, 64:63, 225:224, 2048:2025,...	Pajara	598.45	491.88	2,0	4,−1	3,2	4,2	10.40	3.11
49:48, 81:80, 245:243, 1323:1280,...	Semaphore	1203.67	252.48	1,0	2,−2	4,−8	3,−1	11.20	3.67
81:80, 126:125, 225:224, 3136:3125,...	Meantone	1201.70	504.13	1,0	2,−1	4,−4	7,−10	11.77	1.70
50:49, 81:80, 405:392, 4000:3969,...	Injera	600.89	507.28	2,0	4,−1	8,−4	9,−4	11.92	3.58
49:48, 225:224, 525:512, 686:675,...	Negrisept	1203.19	1078.35	1,0	−2,4	5,−3	1,2	12.12	3.19
64:63, 126:125, 128:125, 4000:3969,...	Augene	399.02	90.59	3,0	5,−1	7,0	8,2	12.13	2.94
49:48, 126:125, 875:864, 1029:1000,...	Keemun	1203.19	317.84	1,0	0,6	1,5	2,3	12.41	3.19
81:80, 128:125, 648:625, 2048:2025,...	Catler	99.81	75.22	12,0	19,0	28,0	33,1	12.84	3.56
50:49, 245:243, 250:243, 2430:2401,...	Hedgehog	598.45	436.13	2,0	1,3	1,5	2,5	13.19	3.11
64:63, 245:243, 1728:1715, 2240:2187,...	Superpyth	1197.60	708.17	1,0	1,1	−3,9	4,−2	14.43	2.40
126:125, 245:243, 686:675, 4375:4374,...	Sensisept	1198.39	755.23	1,0	6,−7	8,−9	11,−13	14.46	1.61
50:49, 525:512, 1029:1024, 1875:1792,...	Lemba	601.70	230.87	2,0	2,3	5,−1	6,−1	14.63	3.74
64:63, 250:243, 875:864, 6144:6125,...	Porcupine	1196.91	1034.59	1,0	−1,3	−2,5	8,−6	14.80	3.09
81:80, 525:512, 875:864, 4375:4374,...	Flattone	1202.54	507.14	1,0	2,−1	4,−4	−1,9	15.38	2.54
225:224, 245:243, 875:864, 3125:3072,...	Magic	1201.28	380.80	1,0	0,5	2,1	−1,12	15.54	1.28
50:49, 875:864, 1728:1715, 3125:3024,...	Doublewide	599.28	326.96	2,0	1,4	3,3	4,3	15.60	3.27
49:48, 250:243, 4000:3969, 6125:5832,...	Nautilus	1202.66	1119.69	1,0	−4,6	−7,10	0,3	15.62	3.48
64:63, 686:675, 2401:2400, 6272:6075,...	Beatles	1197.10	842.38	1,0	3,−2	−4,9	0,4	16.88	2.90
81:80, 686:675, 1029:1000, 10976:10935,...	Liese	1202.62	569.05	1,0	3,−3	8,−12	8,−11	17.49	2.62
81:80, 1029:1024, 1728:1715, 8748:8575,...	Cynder	1201.7	969.18	1,0	4,−3	12,−12	2,1	18.45	1.70
225:224, 1728:1715, 2430:2401, 6144:6125,...	Orwell	1199.53	271.49	1,0	0,7	3,−3	1,8	19.98	0.95
225:224, 3125:3087, 4000:3969, 5120:5103,...	Garibaldi	1200.76	702.64	1,0	1,1	7,−8	11,−14	20.29	0.91
126:125, 1728:1715, 2401:2400, 31104:30625,...	Myna	1198.83	888.94	1,0	9,−10	9,−9	8,−7	20.33	1.17
225:224, 1029:1024, 2401:2400, 16875:16807,...	Miracle	1200.63	116.72	1,0	1,6	3,−7	3,−2	21.10	0.63
A BONUS TEMPERAMENT:									
2401:2400, 4375:4374, 250047:250000,...	Ennealimmal	133.337	84.313	9,0	13,2	19,3	24,2	39.83	0.04

The "main sequence" of Table 1 comprises all possible five-limit 2-D cases where complexity/12 + damage/10 < 1. The "exotemperaments" have larger damage and lend themselves to special measures such as custom inharmonic timbres in order to aid harmoniousness. The "bonus temperaments" are more complex but have exceedingly small damage and sound like JI.

Table 2 is analogous to Table 1 but deals with 2-dimensional temperaments of seven-limit JI. Much of the data was provided by Gene Ward Smith. This case, along with 11-limit and other scenarios, will be discussed more fully in Part 2. Since 7-limit JI is four-dimensional, two independent intervals must vanish in order for a 2-dimensional temperament to result. An infinite number of vanishing intervals can be derived from these two; only the simplest several are shown here. Other than the "bonus temperament," Table 2 comprises all possible seven-limit 2-D cases where complexity/24 + damage/10 < 1. . . . TO BE CONTINUED IN PART 2 . . .

---

[i] See Kyle Gann's "An Introduction to Historical Tunings," http://home.earthlink.net/~kgann/histune.html, which contains only a few minor errors.

[ii] John Chalmers, Paul Hahn, Herman Miller, Manuel Op de Coul, Kees van Prooijen, Margo Schulter, Dan Stearns, and others have enriched this collaboration with their related and often parallel contributions. Crucial germinative interest has been provided by Carl Lumma, Joe Monzo, and Joseph Pehrson.

[iii] For example, Joseph Pehrson has composed in the "Miracle" system (see http://www.soundclick.com/pro/?BandID=104245), Herman Miller in the "Porcupine," "Lemba," and other systems (see http://www.io.com/~hmiller/music/), and Gene Ward Smith in many systems (http://www.xenharmony.org). Though the overwhelming majority of my own public musical output so far has been on standard-tuned instruments, I have procured several microtonal instruments, played them on my own and with the free-improv group MAD DUXX in performances and recordings, and performed sets of alternatively-tuned compositions at two American Festival of Microtonal Music concerts – the first of which was reviewed in Gann, Kyle, June 8, 1999. "Micro Breweries", *The Village Voice* (http://www.villagevoice.com/issues/9922/gann.php) and the second of which had an excerpt broadcast on WNYC-FM on 5/18/01 and 5/31/03 (http://www.wnyc.org/shows/newsounds/episodes/05312003). "Pajara", "Porcupine", and "Superpythagorean" were among the systems described in this paper which were explored in these compositions.

[iv] Though we had worked out many of its details before his appearance in the community, special thanks goes to Gene Ward Smith for applying modern mathematics (namely, the field of Grassmann, or exterior, or multilinear, algebra) to this subject, placing it on a firmer foundation and allowing for many problems to be solved and new results to be obtained.

[v] More precisely, the frequency ratio of a given interval is the *inverse* of its string-length or pipe-length ratio. This is important to keep in mind to avoid misconstruing ancient tuning specifications involving three or more notes.

[vi] There are many ways of understanding this concordance.
The human voice, bowed strings, wind and brass instruments normally produce sounds whose frequency content, or spectrum, consists of exact integer multiples of a fundamental frequency. See Brown, Judith C. 1996. "Frequency ratios of spectral components of musical sounds" *J. Acoust. Soc. Am.* 99, 1210-1218. Such spectra are referred to as *harmonic*. Plucked or hammered strings deviate only slightly from this pattern.
Many of the explanations of the concordance of simple-integer ratios proceed from these facts; see Sethares, W. A. 2005. *Tuning, Timbre, Spectrum, Scale*. Springer-Verlag, London, 2nd edition. These explanations, whose history goes back to Helmholtz, typically consider the psychoacoustic interactions among the spectral components when different notes are combined. Concordance then becomes a *negative* phenomenon which depends on the absence of disturbing interactions. These theories predict that the relative concordance of simple-integer ratios disappears when *inharmonic* instrument spectra – such as those found in Indonesian metallophones, or specially synthesized electronically – are used.

Other explanations begin from the brain's natural propensity to treat harmonic spectra as perceptual gestalts, hearing them as having single rather than multiple pitches. See Parncutt, R. 1989. *Harmony: A Psychoacoustical Approach.*. Springer-Verlag, Berlin. These explanations extend this observation beyond the level of single instrument sounds to the level of harmony, i.e., the combination of notes from several instruments. Concordance here is a *positive* phenomenon which reflects the extent to which combinations of notes can elicit the same gestalt phenomena that apply to the spectra of single notes. The history of such explanations goes back to Rameau. Recent neurological research points to strong evidence for the latter theories and the inadequacy of the Helmholtz-Sethares approach. See Tramo, MJ, Cariani, PA, Delgutte, B, Braida, LD. 2001. "Neurobiological Foundations for the Theory of Harmony in Western Tonal Music." *Annals, New York Academy of Sciences*, vol 92, pp. 92-116 (http://homepage.mac.com/cariani/CarianiWebsite/TramoCarianiNYAS2001.pdf).

[vii] Since the late 15th century, all the thirds in the diatonic scale (C-E, D-F, E-G, F-A, G-B, A-C, and B-D), and most of the fifths (C-G, D-A, E-B, F-C, G-D, and A-E), have been included among its consonant intervals.

[viii] Cents are a common measure of interval size; a ratio n/d has size in cents given by $1200 \cdot \log_2(n/d)$.

[ix] Nevertheless, a couple of the systems in this paper – Helmholtz and Compton – were offered as this kind of solution, avoiding the significant mistunings of meantone temperament, but simplified through what has been called "microtemperament", keeping the overall number of pitches manageable.

[x] Where enharmonic equivalence – the identification of G# with A♭, D# with E♭, C# with D♭, etc. – becomes an explicit and important assumption in the music. See Mathieu, W.A. *Harmonic Experience*. Inner Traditions International, Rochester VM, 1997; also see Kelley, Robert. "Charting Enharmonicism on the Just Intonation *Tonnetz*: A Practical Approach to Neo-Riemannian Analysis (http://garnet.acns.fsu.edu/~rtk1218/justtonnetz.pdf).

[xi] Or F major.

[xii] See the article on Miracle tuning and the decimal keyboard by George Secor in this journal (Xenharmonikôn 18).

[xiii] One may also begin with equal temperaments and combine them into tuning systems of higher dimension, ultimately leading to JI. This is Graham Breed's 'melodic' approach and is the algebraic *dual* of that described in this paper.

[xiv] E.g., 45 factors as 3*3*5; 49 factors as 7*7; so the largest prime factor appearing in 49/45 is 7.

[xv] A vector denoted this way, employing a right-bracket and thus known as a "ket", is a contravariant vector, which is the usual type of vector. See http://mathworld.wolfram.com/ContravariantVector.html. Since the coefficients are restricted to be integers, and not allowed to be other rational or real numbers, etc., we should also add that it is a member of an abelian group or Z-module. See http://mathworld.wolfram.com/Module.html. Thanks to Gene Ward Smith for this note.

[xvi] A covector, or covariant vector, denoted this way, with a left-bracket, is known as a "bra". Strictly speaking, a covariant vector is not a true vector but is a linear functional on a vector space, mapping each vector to a real number. See http://mathworld.wolfram.com/One-Form.html and http://mathworld.wolfram.com/LinearFunctional.html. The particular bra above operates (via the bracket product, or equivalently via the pitch-contour construction) on vectors in the Pythagorean lattice and returns their pitch-differences in cents.

[xvii] See the article on Sagittal notation by George Secor and Dave Keenan in this journal (Xenharmonikôn 18).

[xviii] Prime two is often ignored, because of octave-equivalence, leading to a 2-dimensional lattice. Doing so at this stage leads to mathematical complications when certain temperaments are considered, makes modeling of concordance via lattice proximity more difficult, and does disservice to musicians who don't wish to assume octave-equivalence. Hence, we'll leave prime two in as an autonomous dimension; thus so-called "linear temperaments" will be considered 2-dimensional, etc.

[xix] Tenney, James. "John Cage and the Theory of Harmony", *Soundings* vol. 13, 1984, pp. 55-83.

[xx] The absolute value of a number n, denoted $|n|$, is equal to n if n is positive, or $-n$ if n is negative. The absolute value is never a negative number.

[xxi] Harmonic entropy is a discordance measure I developed based on a minimum of assumptions and parameters. It's described in the 2nd (2005) edition of Sethares, Bill, *Tuning, Timbre, Spectrum, Scale*.

Springer-Verlag, London. However it's closer in spirit to the Tramo, op. cit. view of consonance than to the Sethares view. It's also described and discussed in the internet forum http://groups.yahoo.com/group/harmonic_entropy/. Harmonic entropy can show surprising affinities with Tenney's Harmonic Distance (HD) function, log(n·d), at the simple ratios n/d, but is continuous and equally applicable to irrational intervals.

[xxii]This paper deals with *regular* temperament, in which each prime's tuning is altered in the same way regardless of where it appears in the tuning system. Irregular temperaments can usually be viewed as variations of these and are not essentially different in terms of algebraic or group-theoretic structure. Thus Bach will be mentioned in connection with 12-equal though his actual tuning was more likely an unequal but closed 12-tone "well-temperament".

[xxiii] Each ratio in this list is obtained from its predecessor by dividing by 81/80.

[xxiv] I posted this simple but apparently original idea on Jan. 2, 2004 to http://groups.yahoo.com/group/tuning-math/message/8355.

[xxv] The reader may obtain 0.02 when performing this calculation; this is a result of cumulative rounding errors in the calculations above.

[xxvi] The proof follows from the prime factorization theorem. For all primes p, the maximum damage being T implies that

$$\text{mistuning}(p)/\log(p) <= T$$

so

$$\text{mistuning}(p) <= T\cdot\log(p)$$

for all the primes p. If the factors in a given ratio are 2^ a 3^ b 5^ c . . . (each exponent may either be positive or negative), then the mistuning of the chosen ratio cannot be greater than

$$T\cdot(|a|\cdot\log(2) + |b|\cdot\log(3) + |c|\cdot\log(5) \ldots)$$

since the errors in the primes that make up the chosen ratio, at worst, add up without cancellation to the error in the chosen ratio. The HD of the ratio, meanwhile, is exactly

$$(|a|\cdot\log(2) + |b|\cdot\log(3) + |c|\cdot\log(5) \ldots)$$

So the damage done to the ratio cannot be greater than the second-to-last expression divided by the last expression, i.e., T.

[xxvii] Minimizing maximum "damage" is not the only reasonable criterion for optimally tuning a temperament. One might wish to minimize the maximum discordance over some specified set of 'concordant' intervals. This would put less emphasis on accurately tuning the simplest ratios and more on accurately tuning some more complex ones. And to be fair, increase in discordance is only one kind of audible damage that can be done to a frequency ratio by mistuning it. Some other kinds are increasing beat-rate and "loss of identity", where for example a medium complexity ratio, such as 9/7 or 11/8, loses its special quality and becomes just another badly tuned version of a simpler nearby ratio. Moderately complex ratios are more sensitive to these kinds of damage than simple ratios and so Tenney optimization may not always be appropriate, depending on the desired musical effects. Thanks to Dave Keenan for this note.

[xxviii] Though any sufficiently small tempering of this prime would leave the optimality intact, leaving it just minimizes the damage to intervals that do involve this prime.

[xxix] The reader may obtain -0.02 when performing this calculation; this is a result of cumulative rounding errors in the calculations above.

[xxx] In all cases, it minimizes the maximum inverse-HD-weighted error over all the intervals in the infinite lattice of ratios (or equivalently, any sufficient subset thereof, for example a set of 'consonances'). The calculation for equal temperaments is simple and is illustrated at http://groups.yahoo.com/group/tuning-math/message/8512. The proof in the footnote above applies to all forms of TOP. Thus TOP puts explicit importance on accurately approximating the simplest ratios but does not disregard the effect on the more complex ratios as some other optimizations do.

[xxxi] As a nod to Erv Wilson's Golden Horagrams, I distinguish the DESs where the step sizes are in a ratio smaller than the golden ratio from those where the ratio of the step sizes is larger than the golden ratio. The font in which the DES's cardinality is printed is larger and bolder in the former case than in the latter.

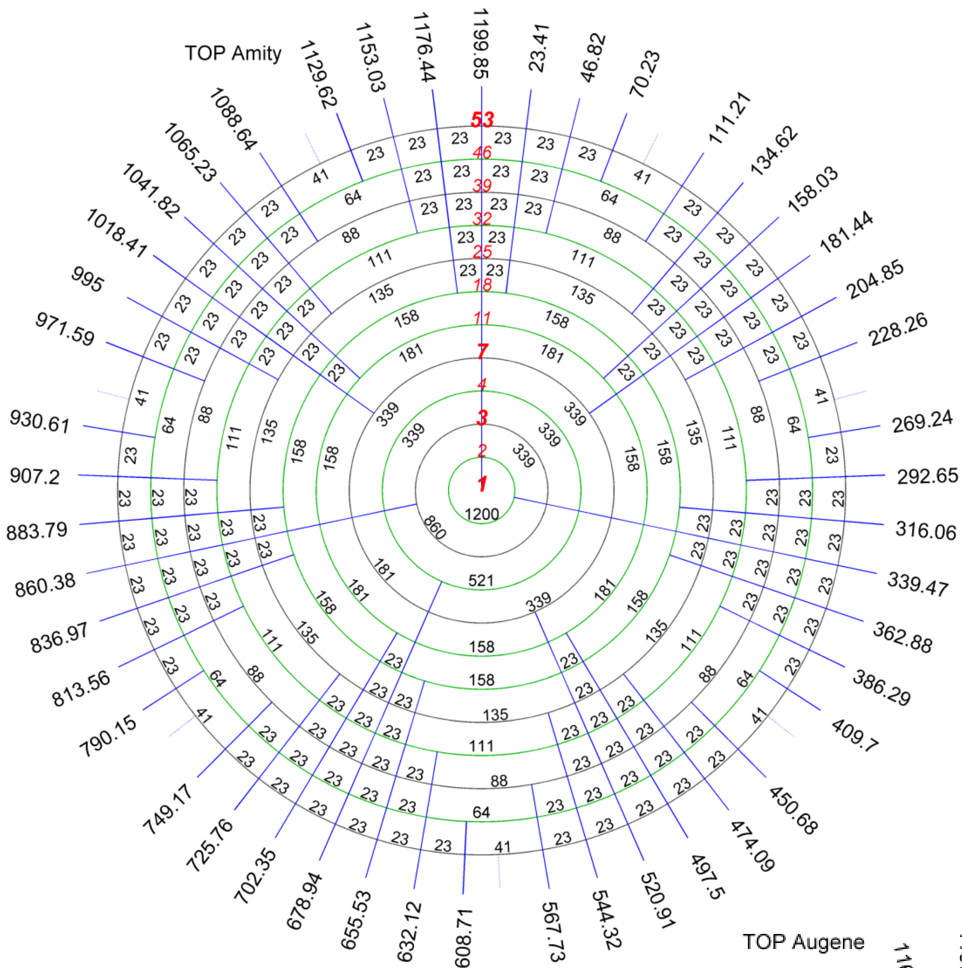
[xxxii] All powers of this ratio will vanish as well; only the simplest vanishing ratio is shown.

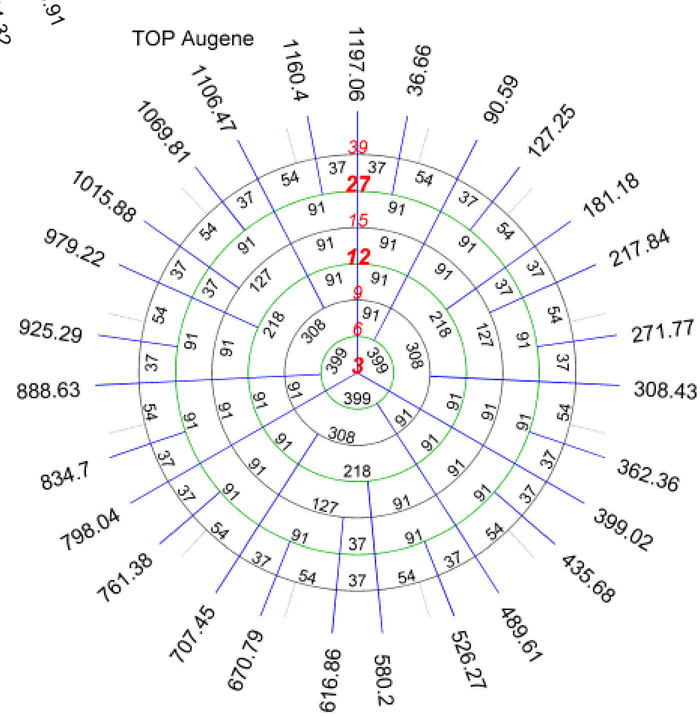
[xxxiii] In the 5-limit case, this is proportional to the Harmonic Distance of the vanishing interval.

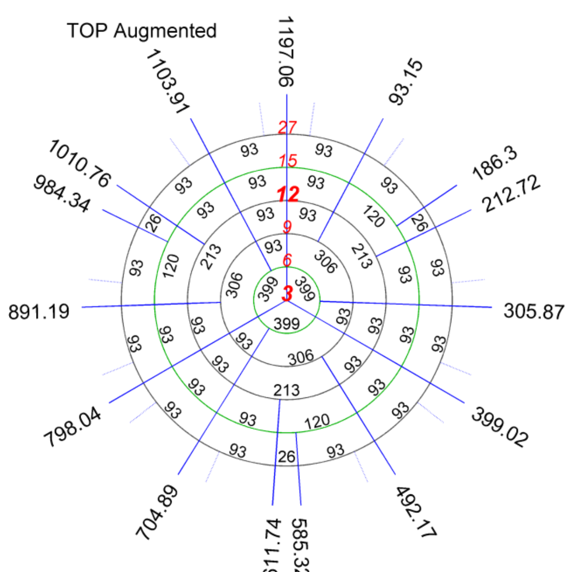
Table 3 -- Approximations of concordant JI intervals in TOP-tuned temperaments. Horagrams' figures contain rounding error; these are more accurate.

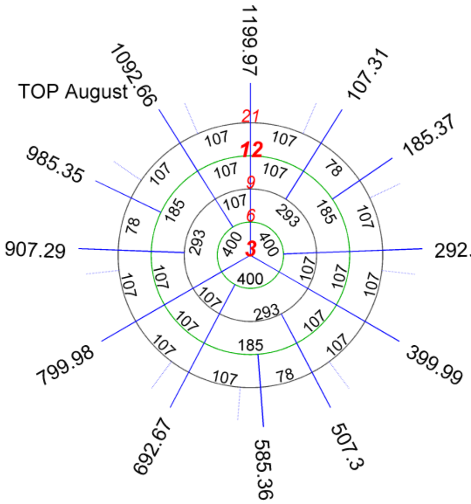
	JI Interval:	ratio: 9/8		ratio: 8/7		ratio: 7/6		ratio: 6/5		ratio: 5/4		ratio: 9/7		ratio: 4/3		ratio: 7/5		ratio: 3/2		ratio: 8/5		ratio: 5/3		ratio: 7/4		ratio: 9/5		ratio: 2/1	
		¢: 203.91		¢: 231.17		¢: 266.87		¢: 315.64		¢: 386.31		¢: 435.08		¢: 498.04		¢: 582.51		¢: 701.96		¢: 813.69		¢: 884.36		¢: 968.83		¢: 1017.60		¢: 1200.00	
		deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl	deviation	tmpr'd intervl
Horagram		(cents)		(cents)		(cents)		(cents)		(cents)		(cents)		(cents)		(cents)		(cents)		(cents)		(cents)		(cents)		(cents)		(cents)	
TOP	Amity	0.9	204.83	#N/A	#N/A	#N/A	#N/A	0.4	316.08	0.0	386.27	#N/A	#N/A	-0.5	497.51	#N/A	#N/A	0.4	702.34	-0.1	813.59	-0.6	883.78	#N/A	#N/A	0.8	1018.42	-0.1	1199.85
TOP	Augene	13.9	217.83	-13.3	217.83	4.9	271.78	-7.2	308.43	12.7	399.02	0.6	435.67	-8.4	489.61	-2.3	580.21	5.5	707.45	-15.6	798.04	4.3	888.63	10.4	979.23	-1.7	1015.87	-2.9	1197.06
TOP	Augmented	8.8	212.73	#N/A	#N/A	#N/A	#N/A	-9.8	305.87	12.7	399.02	#N/A	#N/A	-5.9	492.17	#N/A	#N/A	2.9	704.90	-15.6	798.04	6.8	891.19	#N/A	#N/A	-6.8	1010.77	-2.9	1197.06
TOP	August	-18.5	185.37	-16.6	214.62	25.8	292.68	-23.0	292.68	13.7	399.99	-35.1	399.99	9.3	507.30	2.8	585.36	-9.3	692.67	-13.7	799.98	22.9	907.30	16.5	985.35	-32.2	985.35	0.0	1199.98
TOP	Beatles	17.9	221.78	-9.4	221.78	-1.0	265.89	-5.0	310.61	12.5	398.83	8.5	443.55	-10.4	487.66	-6.0	576.50	7.5	709.44	-15.4	798.27	2.1	886.49	6.5	975.33	2.5	1020.05	-2.9	1197.10
TOP	Blacksmith	35.3	239.18	8.0	239.18	-27.7	239.18	7.4	323.01	8.2	394.53	43.3	478.36	-19.7	478.36	-20.3	562.19	15.6	717.54	-12.3	801.37	-11.5	872.88	-12.1	956.71	22.9	1040.55	-4.1	1195.89
TOP	Blackwood	35.0	238.87	#N/A	#N/A	#N/A	#N/A	3.3	318.96	11.3	397.65	#N/A	#N/A	-20.3	477.73	#N/A	#N/A	14.6	716.60	-17.0	796.69	-9.0	875.38	#N/A	#N/A	18.0	1035.56	-5.7	1194.33
TOP	Bug	-44.9	158.97	#N/A	#N/A	#N/A	#N/A	-55.4	260.26	32.9	419.23	#N/A	#N/A	22.5	520.51	#N/A	#N/A	-22.5	679.49	-32.9	780.77	55.4	939.74	#N/A	#N/A	-77.9	939.74	0.0	1200.00
TOP	Catler	-4.3	199.61	-7.0	224.20	8.0	274.83	-16.2	299.42	12.9	399.22	-11.3	423.81	1.0	499.03	-8.3	574.25	-3.3	698.64	-15.2	798.45	13.9	898.26	4.7	973.48	-19.5	998.06	-2.3	1197.67
TOP	Compton	-3.8	200.10	#N/A	#N/A	#N/A	#N/A	-0.4	315.28	-1.2	385.08	#N/A	#N/A	2.2	500.26	#N/A	#N/A	-1.6	700.36	1.9	815.54	1.0	885.34	#N/A	#N/A	-2.0	1015.64	0.6	1200.62
TOP	Cynder	-10.5	193.43	1.3	232.52	4.7	271.61	-4.9	310.70	0.5	386.86	-9.1	425.95	6.1	504.13	-0.2	582.32	-4.4	697.56	1.2	814.84	6.6	890.99	0.4	969.18	-9.3	1008.27	1.7	1201.70
TOP	Dicot	1.3	205.21	#N/A	#N/A	#N/A	#N/A	37.6	353.22	-33.1	353.22	#N/A	#N/A	3.2	501.22	#N/A	#N/A	4.5	706.43	40.8	854.44	-29.9	854.44	#N/A	#N/A	42.1	1059.65	7.7	1207.66
TOP	Dimipent	-0.6	203.34	#N/A	#N/A	#N/A	#N/A	-16.5	299.16	14.5	400.83	#N/A	#N/A	-1.4	496.65	#N/A	#N/A	-2.0	699.99	-17.9	795.81	13.1	897.48	#N/A	#N/A	-18.4	999.15	-3.4	1196.64
TOP	Dimisept	-1.0	202.91	-34.1	197.08	31.7	298.53	-17.1	298.53	13.7	399.99	-35.1	399.99	-2.4	495.61	14.6	597.06	-3.4	698.52	-19.5	794.14	11.2	895.60	28.2	997.05	-20.5	997.05	-5.9	1194.13
TOP	Dominant	-0.4	203.47	-27.7	203.47	25.5	292.41	-23.2	292.41	20.6	406.93	-28.2	406.93	-2.2	495.88	2.3	584.83	-2.6	699.35	-25.4	788.30	18.5	902.81	22.9	991.76	-25.8	991.76	-4.8	1195.23
TOP	Doublewide	14.7	218.61	-13.5	217.66	5.4	272.31	11.3	326.96	-4.7	381.62	1.2	436.27	-8.1	489.97	16.8	599.28	6.6	708.58	3.3	816.94	-12.8	871.59	12.1	980.89	17.9	1035.55	-1.4	1198.55
TOP	Ennealimmal	0.0	203.92	0.21	231.39	-0.20	266.67	0.06	315.70	-0.04	386.28	0.22	435.30	0.02	498.06	-0.14	582.37	0.02	701.98	0.07	813.76	-0.02	884.34	-0.17	968.65	0.08	1017.68	0.04	1200.04
TOP	Father	87.2	291.10	#N/A	#N/A	#N/A	#N/A	-24.5	291.10	61.1	447.39	#N/A	#N/A	-50.7	447.39	#N/A	#N/A	36.5	738.48	-75.2	738.48	10.4	894.77	#N/A	#N/A	12.0	1029.58	-14.1	1185.87
TOP	Flattone	-15.6	188.26	14.7	245.90	-5.6	261.23	3.2	318.88	-9.8	376.52	-0.9	434.16	9.1	507.14	-2.4	580.11	-6.6	695.40	12.3	826.02	-0.7	883.66	-12.2	956.63	-3.3	1014.28	2.5	1202.54
TOP	Garibaldi	0.6	204.52	-0.3	230.89	0.4	267.23	4.3	319.97	-3.6	382.67	0.3	435.42	0.1	498.12	4.7	587.19	0.7	702.64	4.4	818.09	-3.6	880.79	1.0	969.87	5.0	1022.61	0.8	1200.76
TOP	Hanson	0.0	203.96	#N/A	#N/A	#N/A	#N/A	1.4	317.07	-1.3	385.06	#N/A	#N/A	0.1	498.17	#N/A	#N/A	0.2	702.13	1.5	815.24	-1.1	883.22	#N/A	#N/A	1.6	1019.19	0.3	1200.29
TOP	Hedgehog	19.1	223.00	-18.0	213.13	6.9	273.81	9.0	324.63	-1.0	385.31	1.0	436.13	-11.1	486.95	15.9	598.45	8.0	709.95	-2.1	811.58	-12.1	872.26	14.9	983.76	17.0	1034.58	-3.1	1196.89
TOP	Helmholtz	-0.4	203.51	#N/A	#N/A	#N/A	#N/A	0.1	315.75	-0.3	386.03	#N/A	#N/A	0.2	498.28	#N/A	#N/A	-0.2	701.79	0.3	814.03	0.0	884.31	#N/A	#N/A	-0.1	1017.54	0.1	1200.07
TOP	Injera	-16.7	187.22	-4.7	226.45	14.0	280.83	4.4	320.06	-11.9	374.44	-21.4	413.67	9.2	507.28	18.4	600.89	-7.5	694.50	13.7	827.34	-2.6	881.72	6.5	975.33	-3.0	1014.56	1.8	1201.78
TOP	Keemun	0.5	204.45	18.5	249.68	-17.2	249.68	2.2	317.83	-0.3	385.98	19.1	454.14	1.3	499.37	-15.0	567.52	1.9	703.82	3.5	817.20	1.0	885.35	-15.3	953.50	4.1	1021.65	3.2	1203.19
TOP	Lemba	-22.1	181.85	-0.3	230.87	13.0	279.90	6.2	321.80	-15.5	370.83	-22.4	412.72	12.7	510.78	19.2	601.70	-9.3	692.62	18.9	832.58	-2.8	881.60	3.7	972.53	-3.2	1014.42	3.4	1203.40
TOP	Liese	-10.3	193.58	15.2	246.42	-8.8	258.11	-4.7	310.94	0.8	387.16	4.9	440.00	6.5	504.52	-13.5	569.05	-3.9	698.10	1.8	815.47	7.3	891.68	-12.6	956.21	-8.6	1009.05	2.6	1202.62
TOP	Luna	0.11	204.02	#N/A	#N/A	#N/A	#N/A	-0.03	315.61	0.08	386.39	#N/A	#N/A	-0.07	497.98	#N/A	#N/A	0.05	702.00	-0.10	813.59	0.01	884.37	#N/A	#N/A	0.02	1017.61	-0.02	1199.98
TOP	Magic*	0.2	204.13	*4.4*	*235.56*	*-3.9*	*263.02*	6.3	321.91	-5.5	380.80	*4.6*	*439.69*	0.5	498.57	*2.4*	*584.92*	0.7	702.70	6.8	820.48	-5.0	879.37	*-3.1*	*965.72*	7.0	1024.61	1.3	1201.28
TOP	Mavila	-40.4	163.51	#N/A	#N/A	#N/A	#N/A	11.4	327.02	-28.3	358.01	#N/A	#N/A	23.5	521.52	#N/A	#N/A	-16.9	685.03	34.8	848.54	-4.8	879.53	#N/A	#N/A	-5.6	1012.04	6.5	1206.55
TOP	Meantone*	-10.5	193.43	*3.4*	*234.55*	*2.7*	*269.59*	-4.9	310.70	0.5	386.86	*-7.1*	*427.98*	6.1	504.13	*-2.2*	*580.29*	-4.4	697.56	1.2	814.84	6.6	890.99	*-1.7*	*967.15*	-9.3	1008.27	1.7	1201.70
TOP	Miracle	-3.9	200.02	2.3	233.44	0.0	266.87	1.1	316.74	-2.7	383.59	-1.6	433.46	2.3	500.31	1.1	583.60	-1.6	700.32	3.4	817.04	-0.5	883.89	-1.6	967.19	-0.5	1017.06	0.6	1200.63
TOP	Myna	-0.2	203.71	-2.8	228.41	2.3	269.15	-5.7	309.89	5.1	391.38	-3.0	432.12	-0.5	497.56	-3.5	579.04	-0.7	701.27	-6.2	807.45	4.6	888.94	1.6	970.42	-6.4	1011.16	-1.2	1198.83
TOP	Nautilus	3.1	206.96	17.7	248.92	-17.9	248.92	16.3	331.90	-13.4	372.91	20.8	455.89	-0.2	497.85	-1.7	580.82	2.9	704.81	16.1	829.75	-13.6	870.76	-15.1	953.74	19.1	1036.71	2.7	1202.66
TOP	Negripent	-11.2	192.66	#N/A	#N/A	#N/A	#N/A	3.2	318.81	-7.9	378.44	#N/A	#N/A	6.5	504.58	#N/A	#N/A	-4.7	697.24	9.7	823.39	-1.3	883.02	#N/A	#N/A	-1.5	1016.05	1.8	1201.82
TOP	Negrisept	0.5	204.45	18.5	249.68	-17.2	249.68	13.7	329.29	-11.8	374.53	19.1	454.14	1.3	499.37	-3.5	578.98	1.9	703.82	15.0	828.66	-10.5	873.89	-15.3	953.50	15.5	1033.11	3.2	1203.19
TOP	Orson	-1.5	202.43	#N/A	#N/A	#N/A	#N/A	0.4	316.06	-1.0	385.28	#N/A	#N/A	0.9	498.90	#N/A	#N/A	-0.6	701.34	1.3	814.96	-0.2	884.18	#N/A	#N/A	-0.2	1017.39	0.2	1200.24
TOP	Orwell	-1.6	202.31	-4.1	227.12	4.6	271.49	0.2	315.87	-1.3	385.05	-5.7	429.43	0.6	498.61	4.9	587.36	-1.0	700.92	0.8	814.48	-0.7	883.66	3.6	972.42	-0.8	1016.79	-0.5	1199.53
TOP	Pajara	9.2	213.13	-18.0	213.13	11.9	278.75	4.1	319.70	-1.0	385.31	-8.8	426.27	-6.2	491.88	15.9	598.45	3.1	705.01	-2.1	811.58	-7.2	877.19	14.9	983.76	7.1	1024.71	-3.1	1196.89
TOP	Passion	10.4	214.31	#N/A	#N/A	#N/A	#N/A	-2.9	312.71	7.3	393.60	#N/A	#N/A	-6.0	492.00	#N/A	#N/A	4.4	706.31	-9.0	804.71	1.2	885.60	#N/A	#N/A	1.4	1019.03	-1.7	1198.31
TOP	Porcupine*	19.1	223.00	*-8.2*	*223.00*	*-2.9*	*263.95*	9.0	324.64	-1.0	385.32	*10.9*	*446.00*	-11.1	486.95	*6.1*	*588.59*	8.0	709.95	-2.1	811.59	-12.1	872.27	*5.1*	*973.91*	17.0	1034.59	-3.1	1196.91
TOP	Ripple	-20.5	183.40	#N/A	#N/A	#N/A	#N/A	-9.7	305.98	1.1	387.38	#N/A	#N/A	11.9	509.96	#N/A	#N/A	-8.6	693.36	2.3	815.94	13.0	897.35	#N/A	#N/A	-18.3	999.34	3.3	1203.32
TOP	Semaphore	-10.2	193.75	21.3	252.48	-14.4	252.48	-4.4	311.21	1.2	387.49	11.1	446.23	6.9	504.96	-18.8	563.69	-3.2	698.71	2.5	816.17	8.1	892.46	-17.6	951.19	-7.7	1009.92	3.7	1203.67
TOP	Sensipent	-0.1	203.84	#N/A	#N/A	#N/A	#N/A	-2.0	313.62	1.8	388.09	#N/A	#N/A	-0.2	497.87	#N/A	#N/A	-0.2	701.71	-2.2	811.49	1.6	885.97	#N/A	#N/A	-2.3	1015.33	-0.4	1199.59
TOP	Sensisept	8.4	212.30	-0.3	230.86	-4.7	262.18	-3.6	312.07	7.0	393.27	8.1	443.16	-5.0	493.05	-8.3	574.25	3.4	705.34	-8.6	805.12	2.0	886.32	-1.3	967.53	-0.2	1017.41	-1.6	1198.39
TOP	Srutal	5.5	209.40	#N/A	#N/A	#N/A	#N/A	-1.5	314.10	3.8	390.16	#N/A	#N/A	-3.2	494.86	#N/A	#N/A	2.3	704.25	-4.7	808.95	0.7	885.01	#N/A	#N/A	0.8	1018.35	-0.9	1199.11
TOP	Superpyth*	14.8	218.74	*###*	*218.74*	*3.8*	*270.69*	7.0	322.63	-0.8	385.54	*2.4*	*437.48*	-8.6	489.43	*10.8*	*593.31*	6.2	708.17	-1.6	812.06	-9.4	874.97	*10.0*	*978.85*	13.2	1030.80	-2.4	1197.60
TOP	Tetracot	6.0	209.89	#N/A	#N/A	#N/A	#N/A	2.8	318.46	-0.3	386.00	#N/A	#N/A	-3.5	494.57	#N/A	#N/A	2.5	704.46	-0.7	813.03	-3.8	880.57	#N/A	#N/A	5.3	1022.92	-1.0	1199.03
TOP	Vishnu	-0.01	203.90	#N/A	#N/A	#N/A	#N/A	-0.25	315.39	0.22	386.54	#N/A	#N/A	-0.02	498.02	#N/A	#N/A	-0.03	701.92	-0.27	813.41	0.20	884.56	#N/A	#N/A	-0.28	1017.31	-0.05	1199.95
TOP	Würschmidt	-0.1	203.86	#N/A	#N/A	#N/A	#N/A	-1.5	314.13	1.3	387.64	#N/A	#N/A	-0.1	497.92	#N/A	#N/A	-0.2	701.77	-1.6	812.05	1.2	885.56	#N/A	#N/A	-1.7	1015.90	-0.3	1199.69

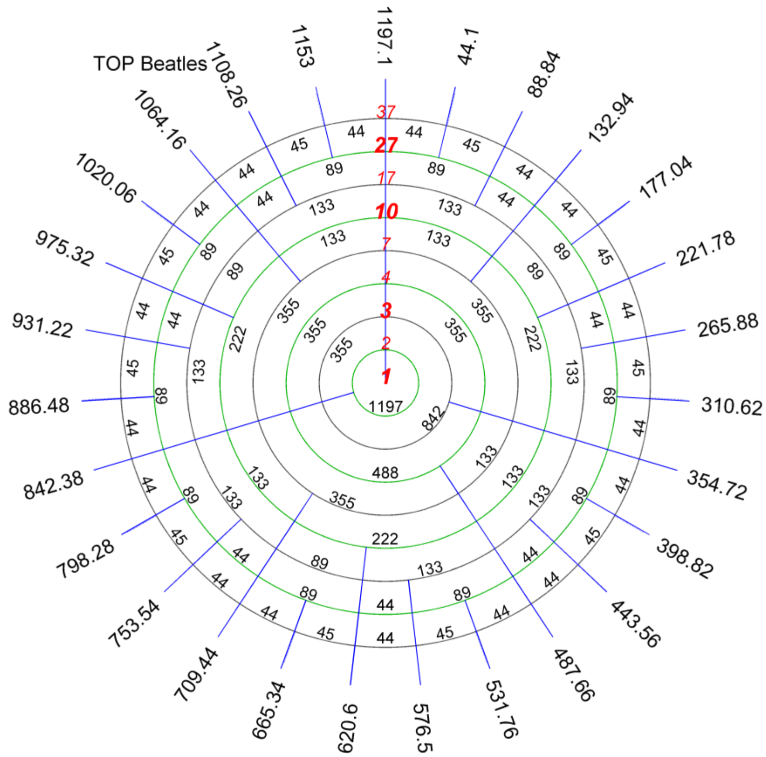
*figures in italics should be ignored in 5-limit context

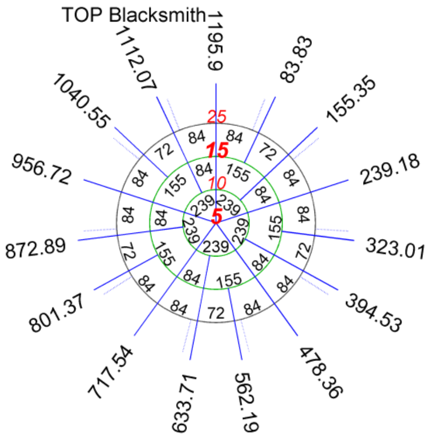

TOP Amity
1199.85
23.41
46.82
70.23
111.21
134.62
158.03
181.44
204.85
228.26
269.24
292.65
316.06
339.47
362.88
386.29
409.7
450.68
474.09
497.5
520.91
544.32
567.73
608.71
632.12
655.53
678.94
702.35
725.76
749.17
790.15
813.56
836.97
860.38
883.79
907.2
930.61
971.59
995
1018.41
1041.82
1065.23
1088.64
1129.62
1153.03
1176.44

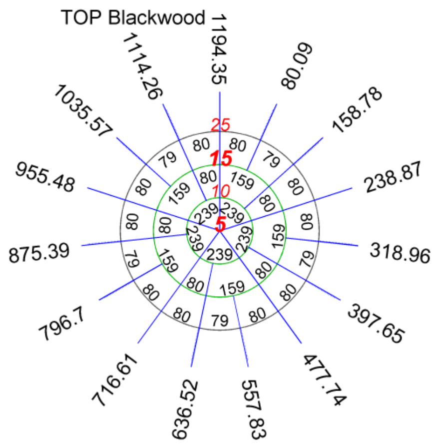

TOP Augene
1197.06
36.66
90.59
127.25
181.18
217.84
271.77
308.43
362.36
399.02
435.68
489.61
526.27
580.2
616.86
670.79
707.45
761.38
798.04
834.7
888.63
925.29
979.22
1015.88
1069.81
1106.47
1160.4

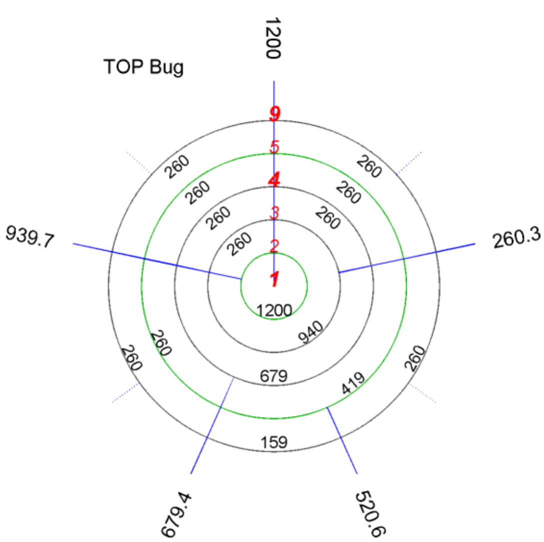

TOP Augmented
1197.06
93.15
186.3
212.72
305.87
399.02
492.17
585.32
611.74
704.89
798.04
891.19
984.34
1010.76
1103.91

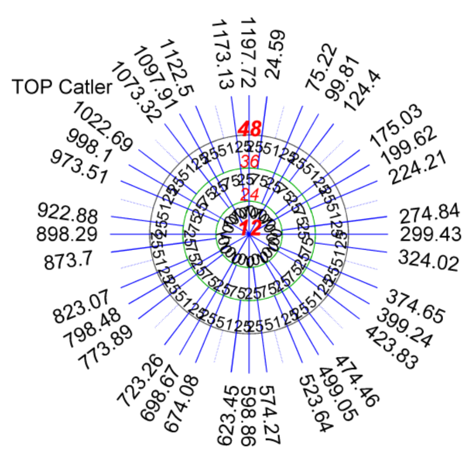

TOP August
1199.97
107.31
185.37
292.68
399.99
507.3
585.36
692.67
799.98
907.29
985.35
1092.66

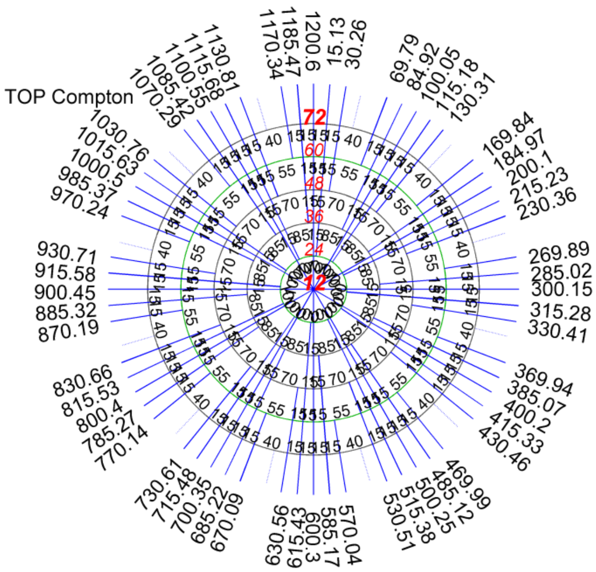

TOP Beatles
1197.1
44.1
88.84
132.94
177.04
221.78
265.88
310.62
354.72
398.82
443.56
487.66
531.76
576.5
620.6
665.34
709.44
753.54
798.28
842.38
886.48
931.22
975.32
1020.06
1064.16
1108.26
1153

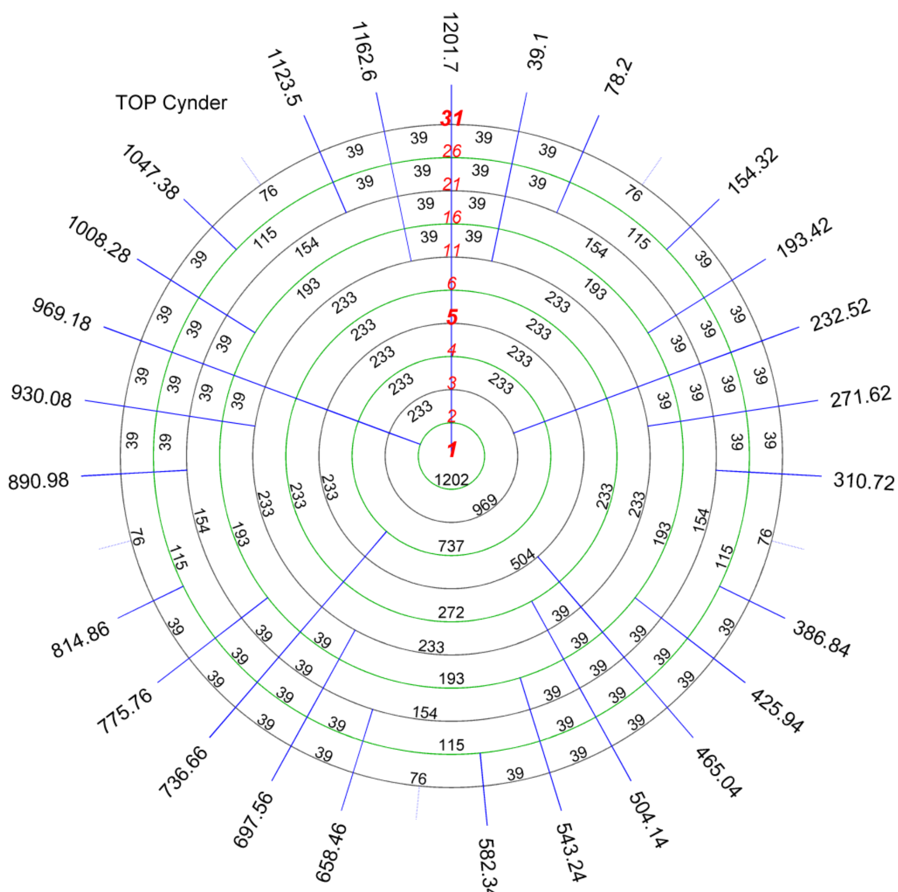

TOP Blacksmith
1195.9
83.83
155.35
239.18
323.01
394.53
478.36
562.19
633.71
717.54
801.37
872.89
956.72
1040.55
1112.07

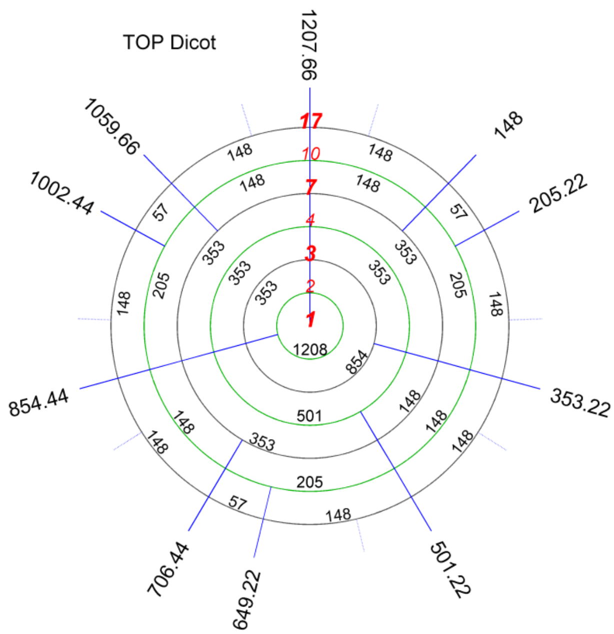

TOP Blackwood
1194.35
80.09
158.78
238.87
318.96
397.65
477.74
557.83
636.52
716.61
796.7
875.39
955.48
1035.57
1114.26

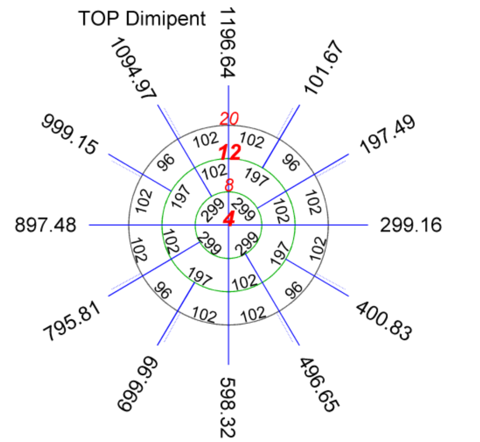

TOP Bug
1200
260.3
520.6
679.4
939.7

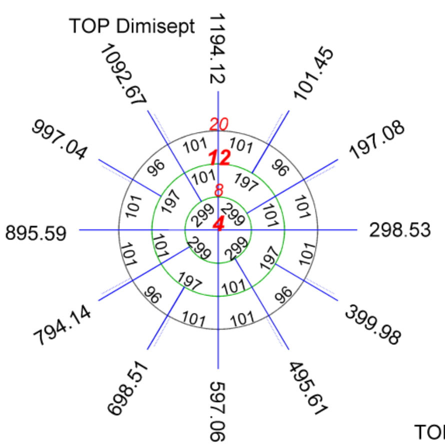

TOP Catler
1197.72
24.59
75.22
99.81
124.4
175.03
199.62
224.21
274.84
299.43
324.02
374.65
399.24
423.83
474.46
499.05
523.64
574.27
598.86
623.45
674.08
698.67
723.26
773.89
798.48
823.07
873.7
898.29
922.88
973.51
998.1
1022.69
1073.32
1097.91
1122.5
1173.13

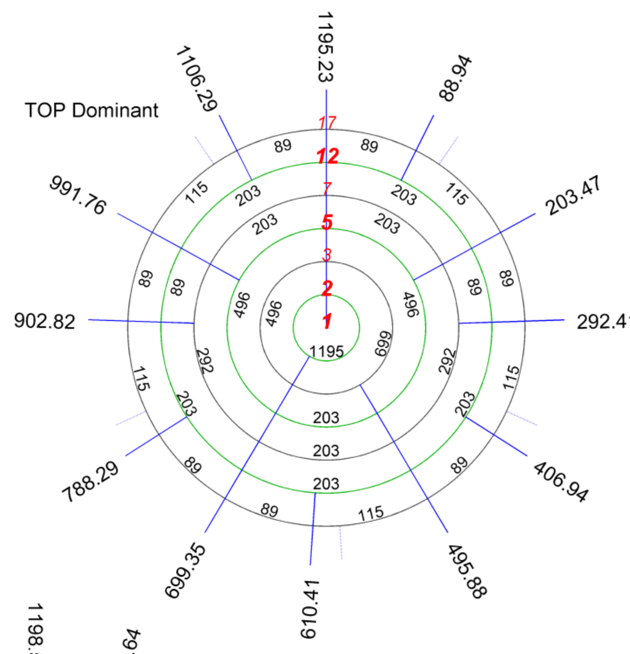

TOP Compton
1170.34
1185.47
1200.6
15.13
30.26
69.79
84.92
100.05
115.18
130.31
169.84
184.97
200.1
215.23
230.36
269.89
285.02
300.15
315.28
330.41
369.94
385.07
400.2
415.33
430.46
469.99
485.12
500.25
515.38
530.51
570.04
585.17
600.3
615.43
630.56
670.09
685.22
700.35
715.48
730.61
770.14
785.27
800.4
815.53
830.66
870.19
885.32
900.45
915.58
930.71
970.24
985.37
1000.5
1015.63
1030.76
1070.29
1085.42
1100.55
1115.68
1130.81
72
60
48
36
24
12

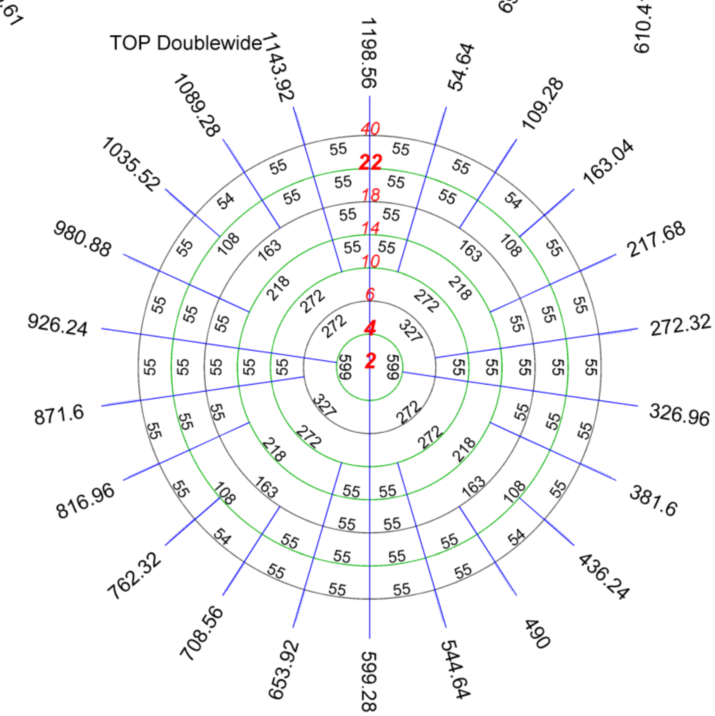

TOP Cynder
1201.7
39.1
78.2
154.32
193.42
232.52
271.62
310.72
386.84
425.94
465.04
504.14
543.24
582.34
658.46
697.56
736.66
775.76
814.86
890.98
930.08
969.18
1008.28
1047.38
1123.5
1162.6
31
26
21
16
11
6
5
4
3
2
1
1202
969
737
504
272
233
193
154
115
76
39

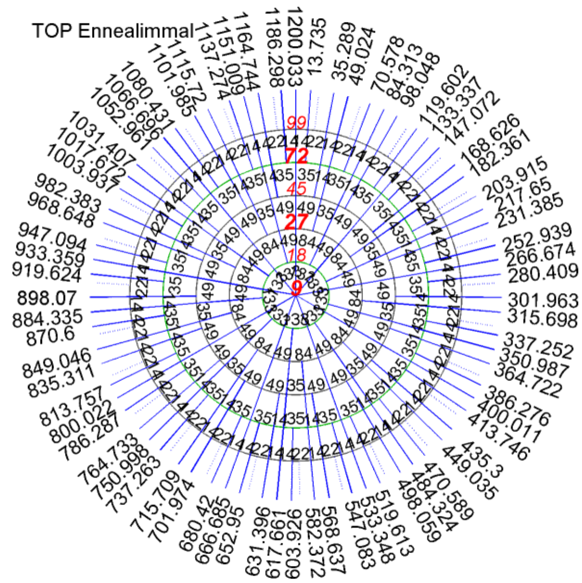

TOP Dicot
1207.66
148
205.22
353.22
501.22
649.22
706.44
854.44
1002.44
1059.66
17
10
7
4
3
2
1
1208
854
501
353
205
148
57

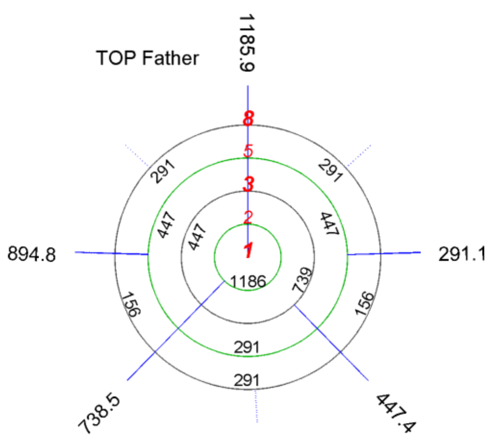

TOP Dimipent
1196.64
101.67
197.49
299.16
400.83
496.65
598.32
699.99
795.81
897.48
999.15
1094.97
20
12
8
4

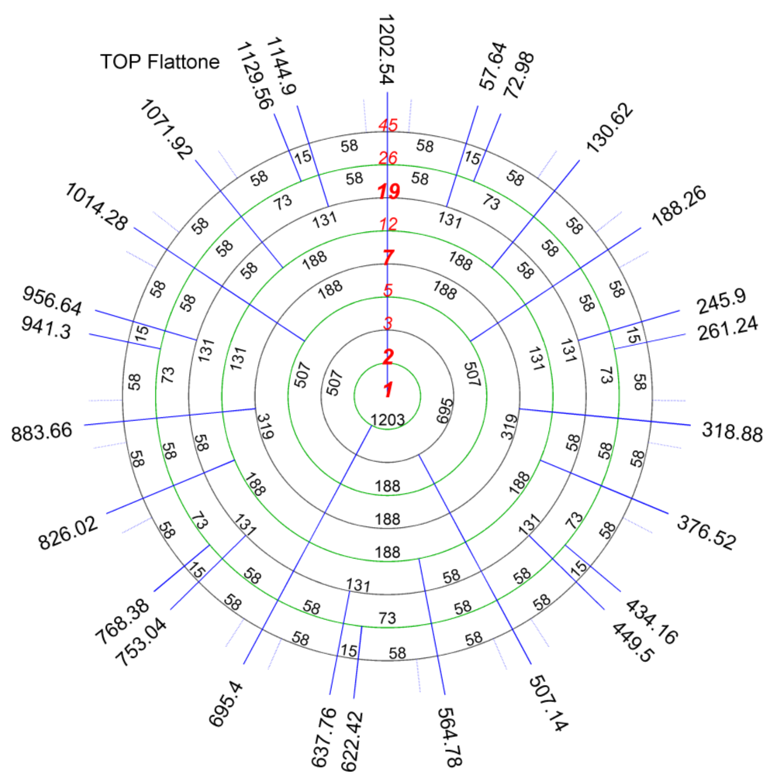

TOP Dimisept
1194.12
101.45
197.08
298.53
399.98
495.61
597.06
698.51
794.14
895.59
997.04
1092.67
20
12
8
4


TOP Dominant
1195.23
88.94
203.47
292.41
406.94
495.88
610.41
699.35
788.29
902.82
991.76
1106.29
17
12
7
5
3
2
1
1195

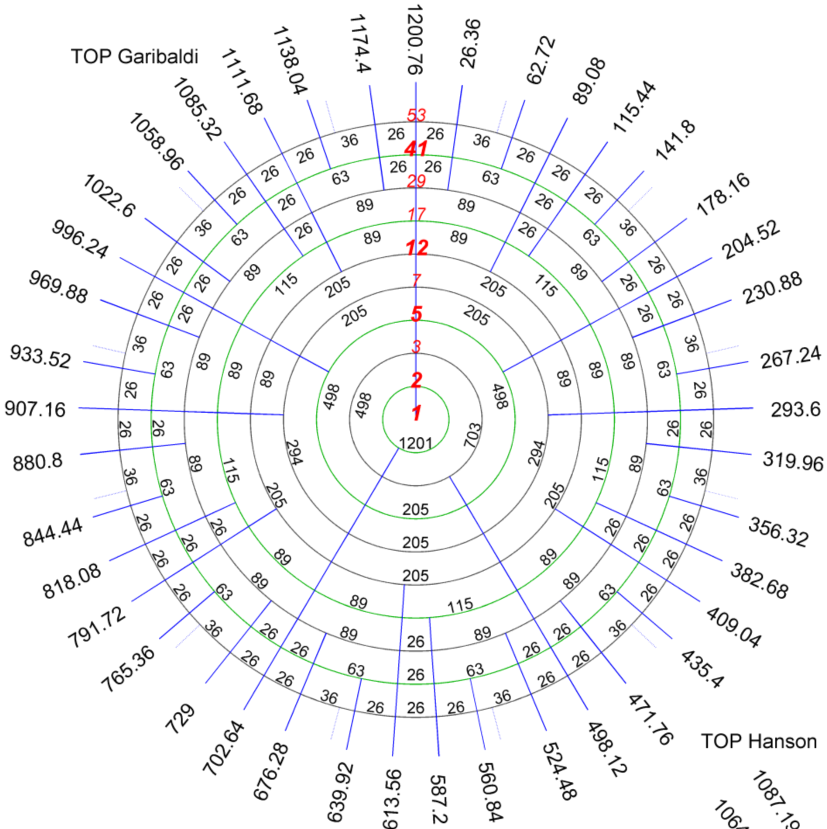

TOP Doublewide
1198.56
54.64
109.28
163.04
217.68
272.32
326.96
381.6
436.24
490
544.64
599.28
653.92
708.56
762.32
816.96
871.6
926.24
980.88
1035.52
1089.28
1143.92
40
22
18
14
10
6
4
2


TOP Ennealimmal
1200.033
13.735
35.289
49.024
70.578
84.313
98.048
119.602
133.337
147.072
168.626
182.361
203.915
217.65
231.385
252.939
266.674
280.409
301.963
315.698
337.252
350.987
364.722
386.276
400.011
413.746
435.3
449.035
470.589
484.324
498.059
519.613
533.348
547.083
568.637
582.372
603.926
617.661
631.396
652.95
666.685
680.42
701.974
715.709
737.263
750.998
764.733
786.287
800.022
813.757
835.311
849.046
870.6
884.335
898.07
919.624
933.359
947.094
968.648
982.383
1003.937
1017.672
1031.407
1052.961
1066.696
1080.431
1101.985
1115.72
1137.274
1151.009
1164.744
1186.298
99
72
45
27
18
9

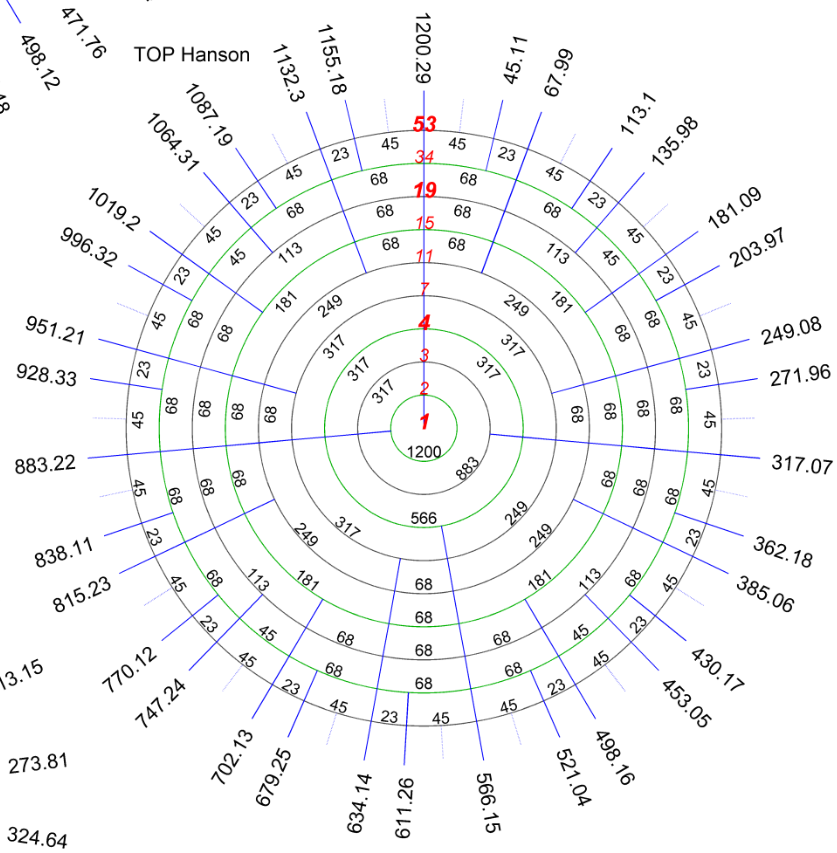

TOP Father
1185.9
291.1
447.4
738.5
894.8
8
5
3
2
1
291
447
1186
739
156


TOP Flattone
1202.54
57.64
72.98
130.62
188.26
245.9
261.24
318.88
376.52
434.16
449.5
507.14
564.78
622.42
637.76
695.4
753.04
768.38
826.02
883.66
941.3
956.64
1014.28
1071.92
1129.56
1144.9
45
26
19
12
7
5
3
2
1
1203

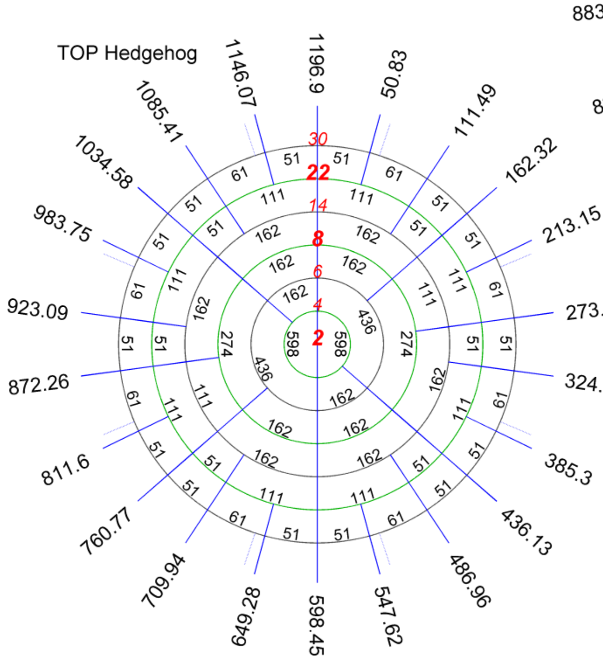
TOP Garibaldi

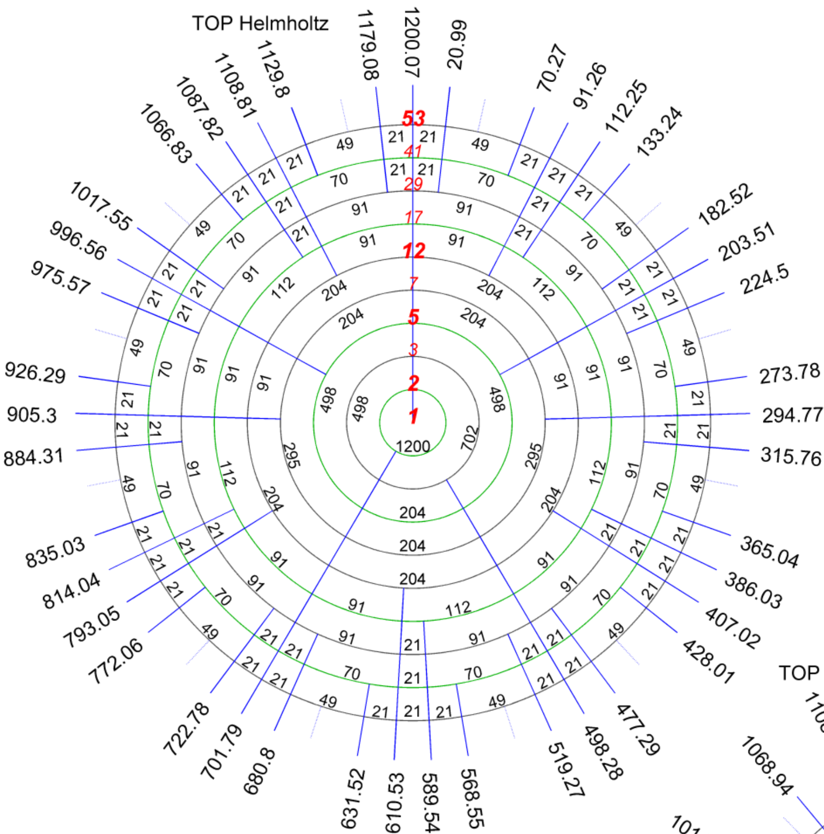


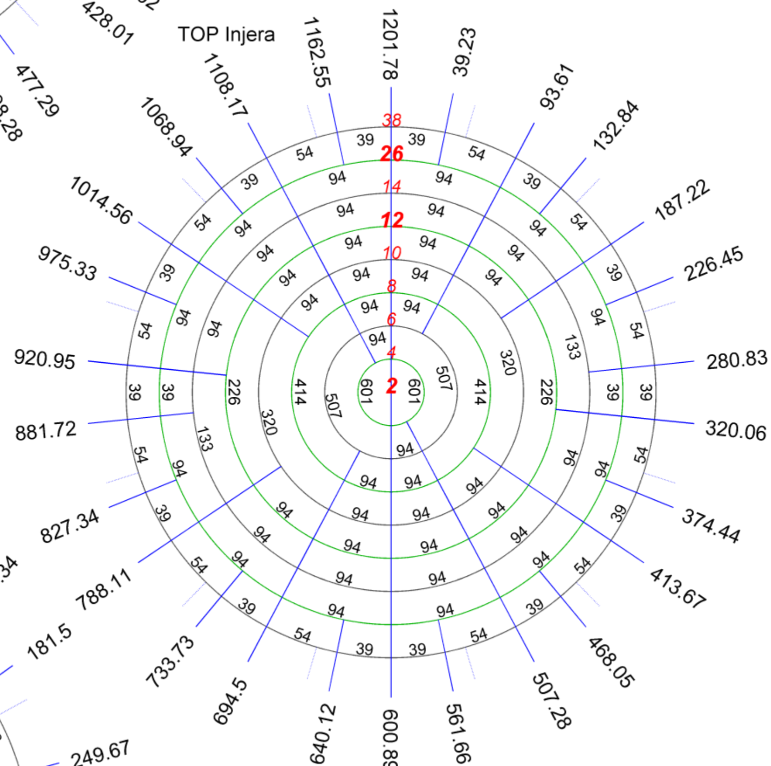
TOP Hanson

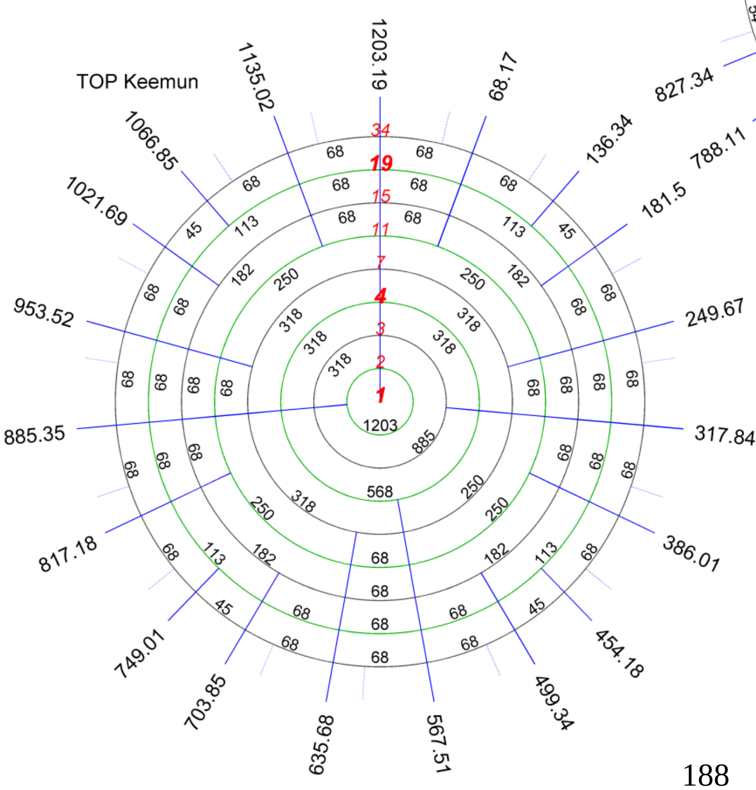


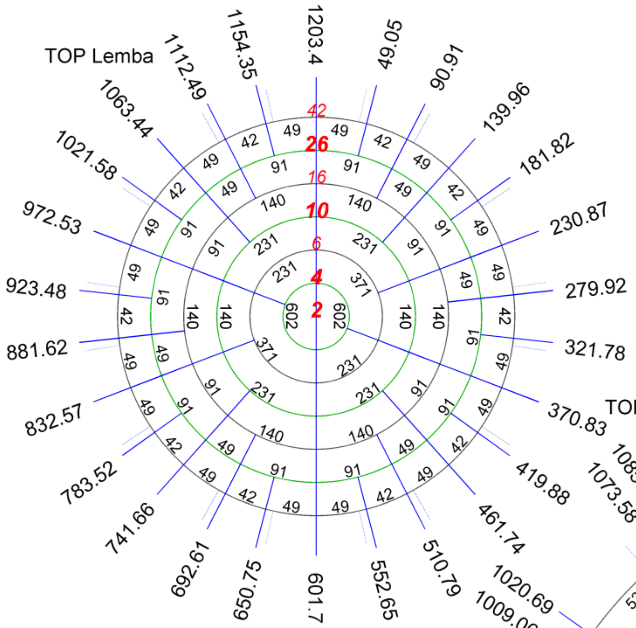
TOP Hedgehog

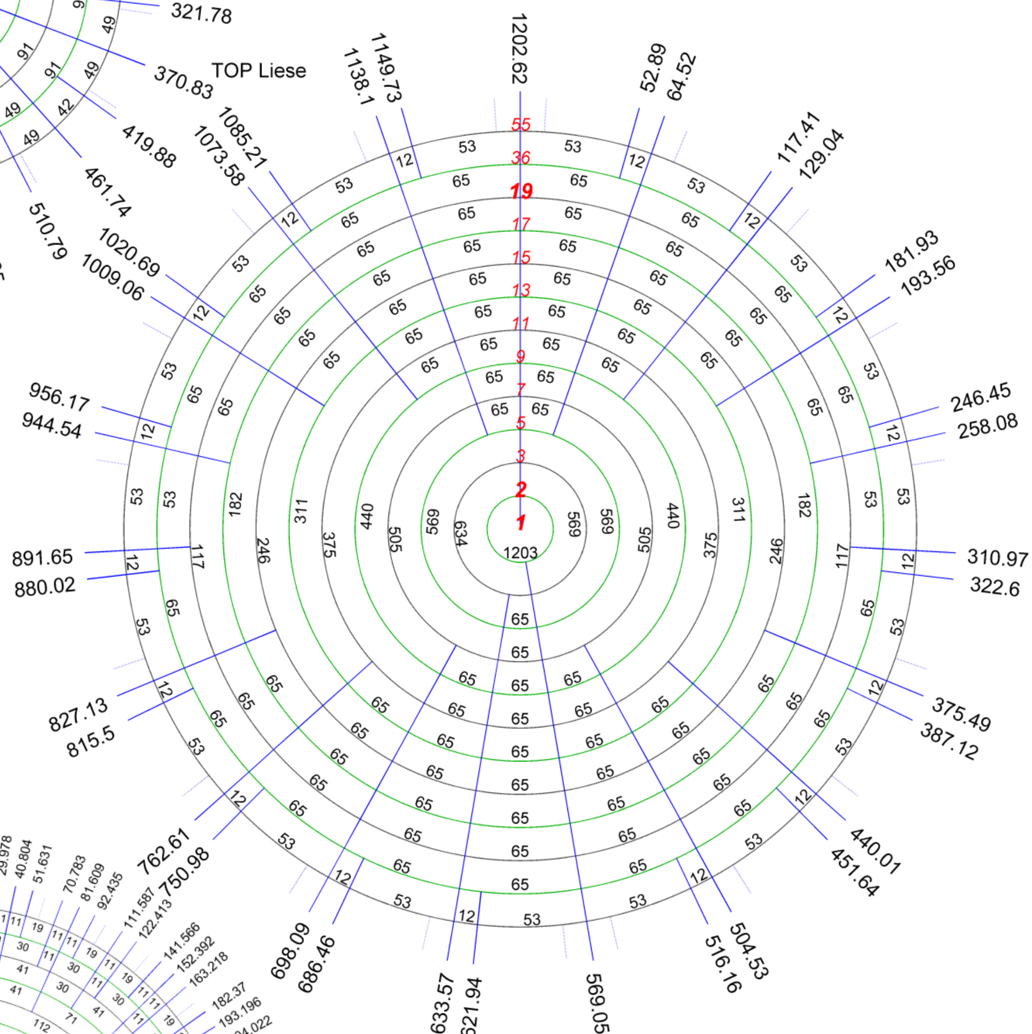


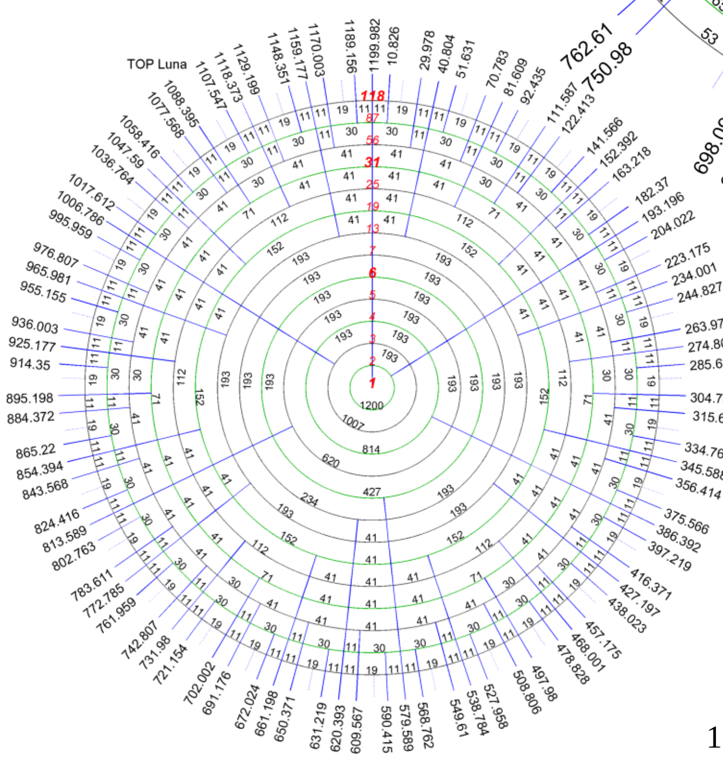

TOP Helmholtz
1200.07
20.99
70.27
91.26
112.25
133.24
182.52
203.51
224.5
273.78
294.77
315.76
365.04
386.03
407.02
428.01
477.29
498.28
519.27
568.55
589.54
610.53
631.52
680.8
701.79
722.78
772.06
793.05
814.04
835.03
884.31
905.3
926.29
975.57
996.56
1017.55
1066.83
1087.82
1108.81
1129.8
1179.08

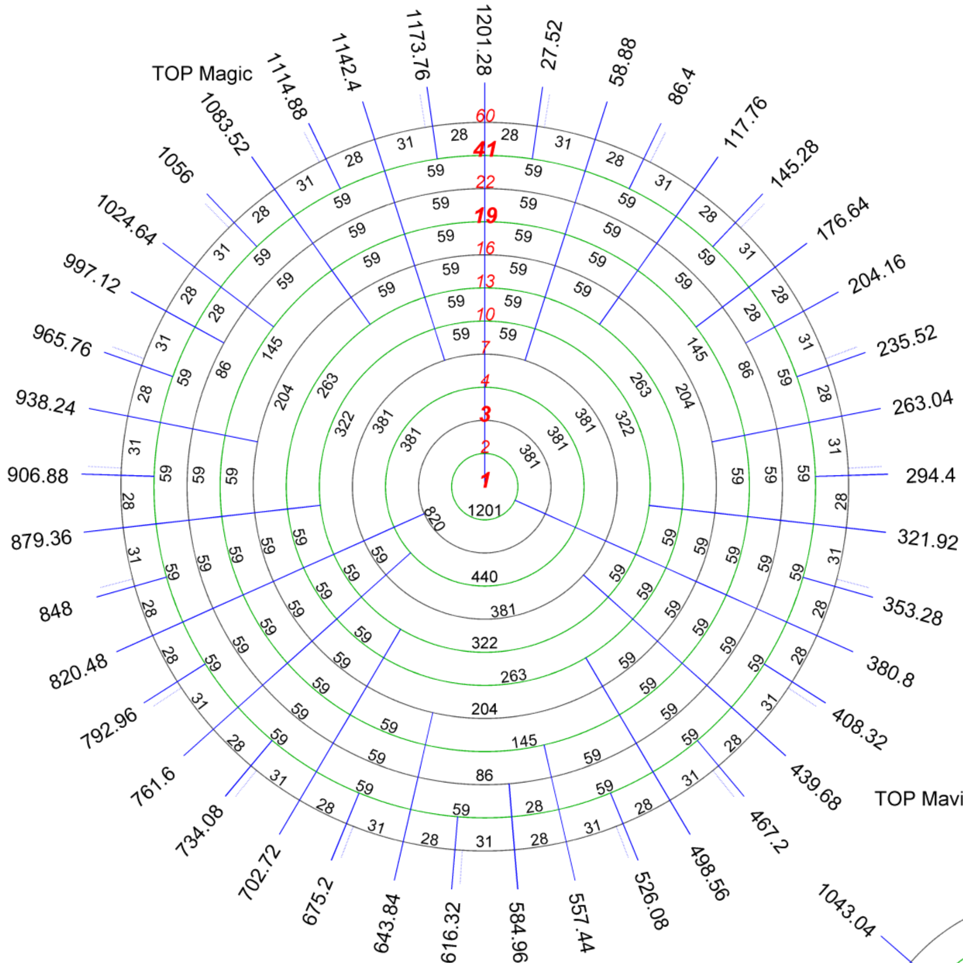

TOP Injera
1201.78
39.23
93.61
132.84
187.22
226.45
280.83
320.06
374.44
413.67
468.05
507.28
561.66
600.89
640.12
694.5
733.73
788.11
827.34
881.72
920.95
975.33
1014.56
1068.94
1108.17
1162.55

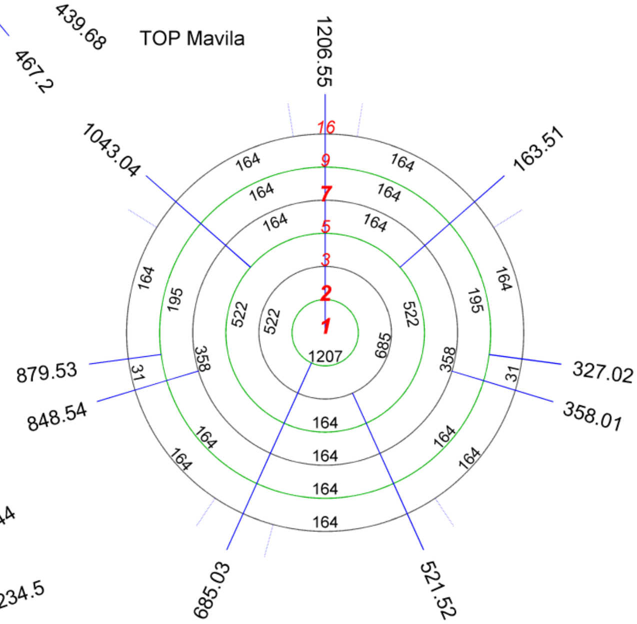

TOP Keemun
1203.19
68.17
136.34
181.5
249.67
317.84
386.01
454.18
499.34
567.51
635.68
703.85
749.01
817.18
885.35
953.52
1021.69
1066.85
1135.02

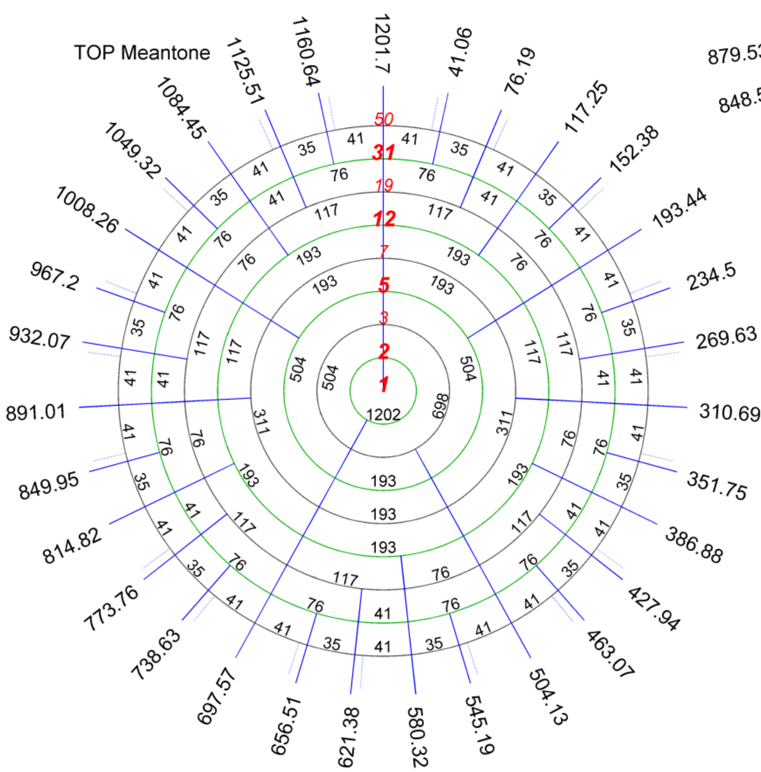

TOP Lemba
1203.4
49.05
90.91
139.96
181.82
230.87
279.92
321.78
370.83
419.88
461.74
510.79
552.65
601.7
650.75
692.61
741.66
783.52
832.57
881.62
923.48
972.53
1021.58
1063.44
1112.49
1154.35


TOP Liese
1202.62
52.89
64.52
117.41
129.04
181.93
193.56
246.45
258.08
310.97
322.6
375.49
387.12
440.01
451.64
504.53
516.16
569.05
621.94
633.57
686.46
698.09
750.98
762.61
815.5
827.13
880.02
891.65
944.54
956.17
1009.06
1020.69
1073.58
1085.21
1138.1
1149.73

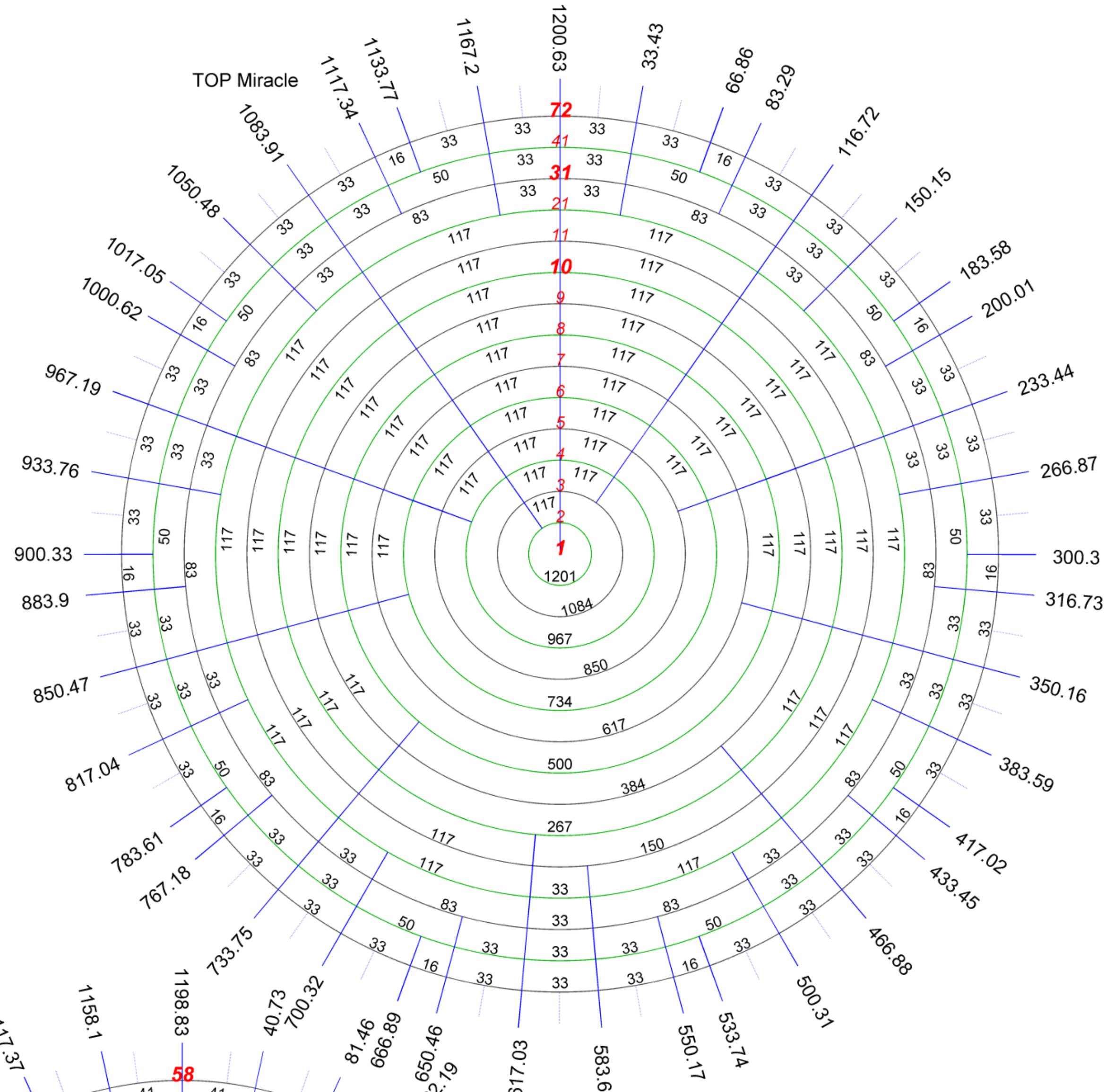

TOP Luna


TOP Magic
1201.28
27.52
58.88
86.4
117.76
145.28
176.64
204.16
235.52
263.04
294.4
321.92
353.28
380.8
408.32
439.68
467.2
498.56
526.08
557.44
584.96
616.32
643.84
675.2
702.72
734.08
761.6
792.96
820.48
848
879.36
906.88
938.24
965.76
997.12
1024.64
1056
1083.52
1114.88
1142.4
1173.76

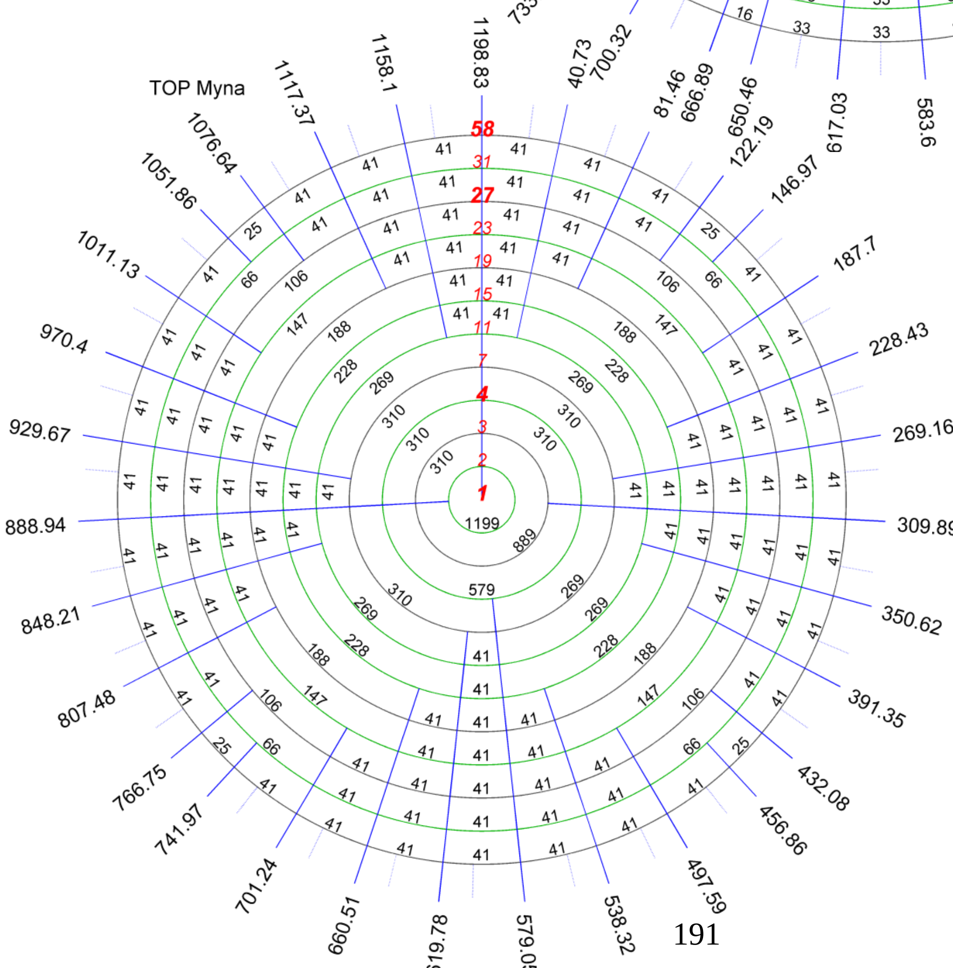

TOP Mavila
1206.55
163.51
327.02
358.01
521.52
685.03
848.54
879.53
1043.04

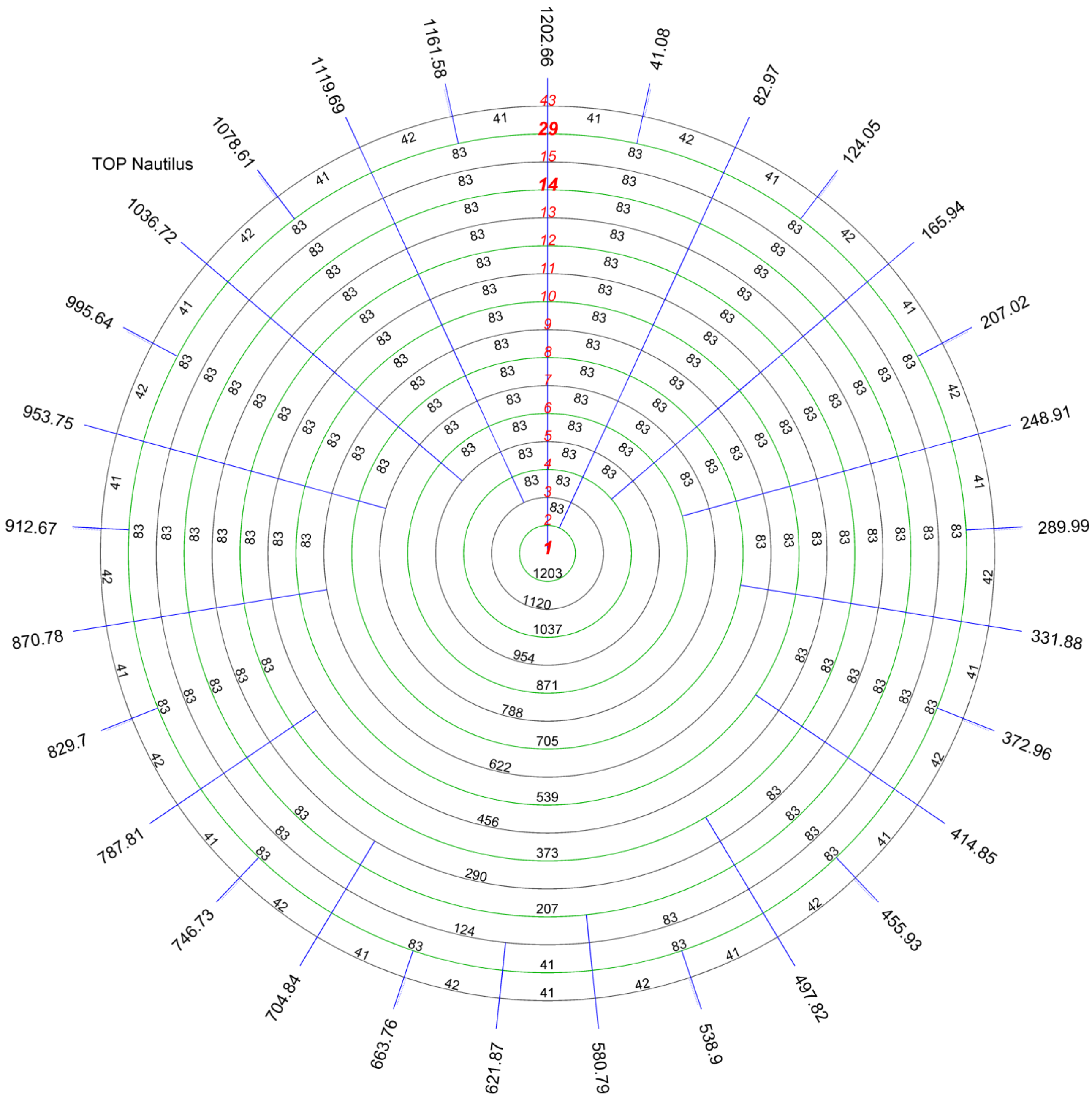

TOP Meantone
1201.7
41.06
76.19
117.25
152.38
193.44
234.5
269.63
310.69
351.75
386.88
427.94
463.07
504.13
545.19
580.32
621.38
656.51
697.57
738.63
773.76
814.82
849.95
891.01
932.07
967.2
1008.26
1049.32
1084.45
1125.51
1160.64

TOP Miracle

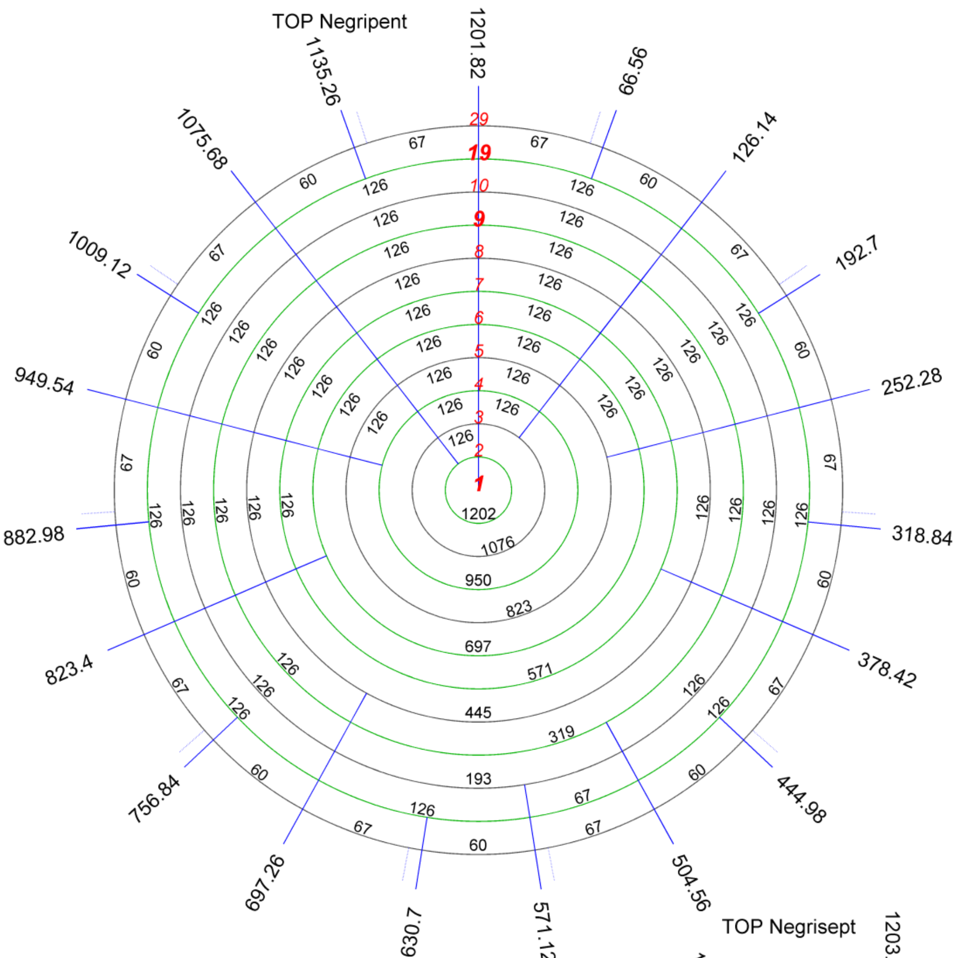


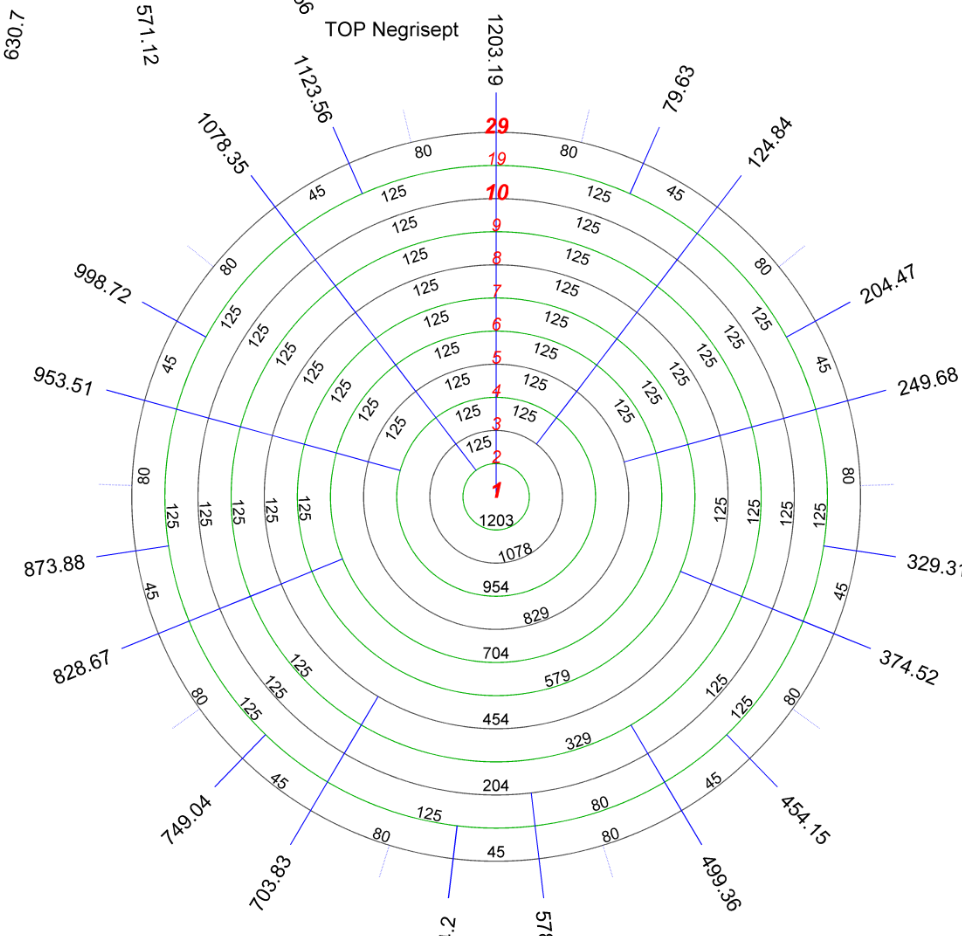
TOP Myna

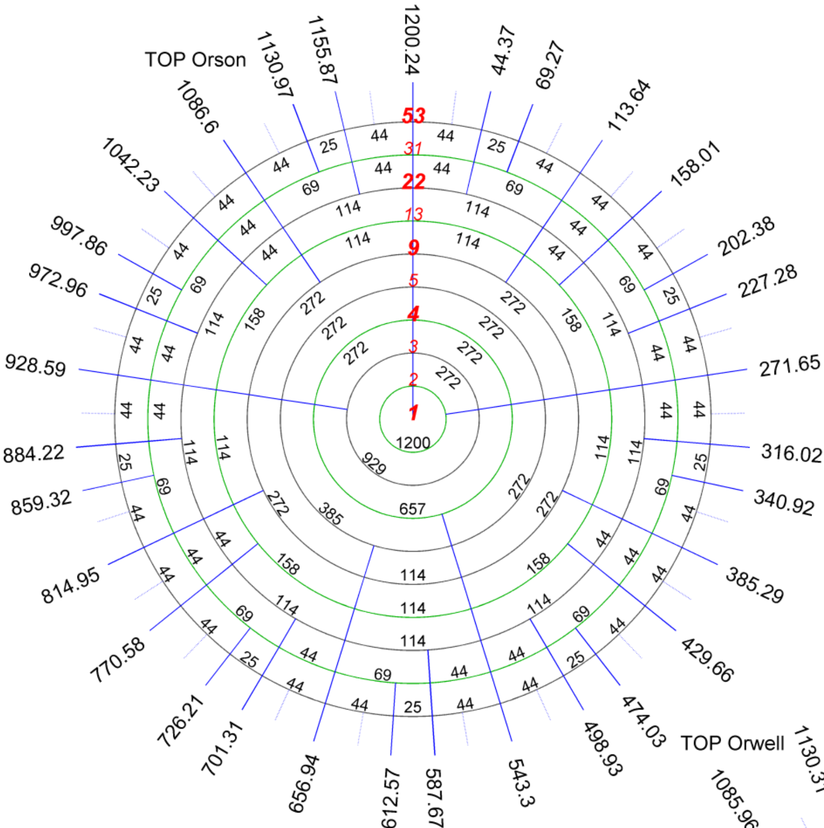


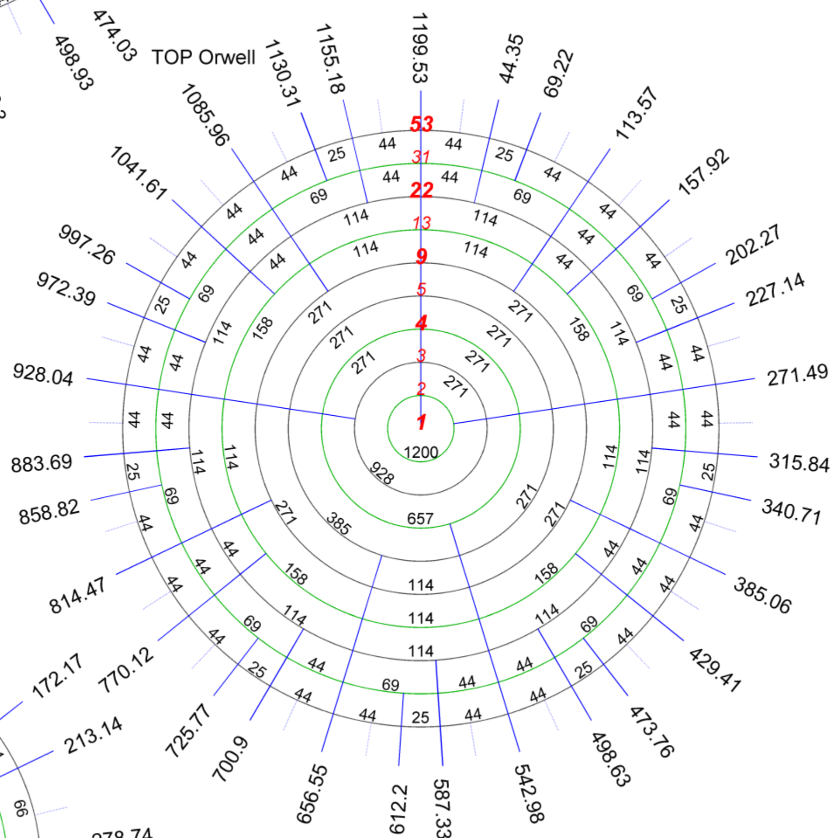


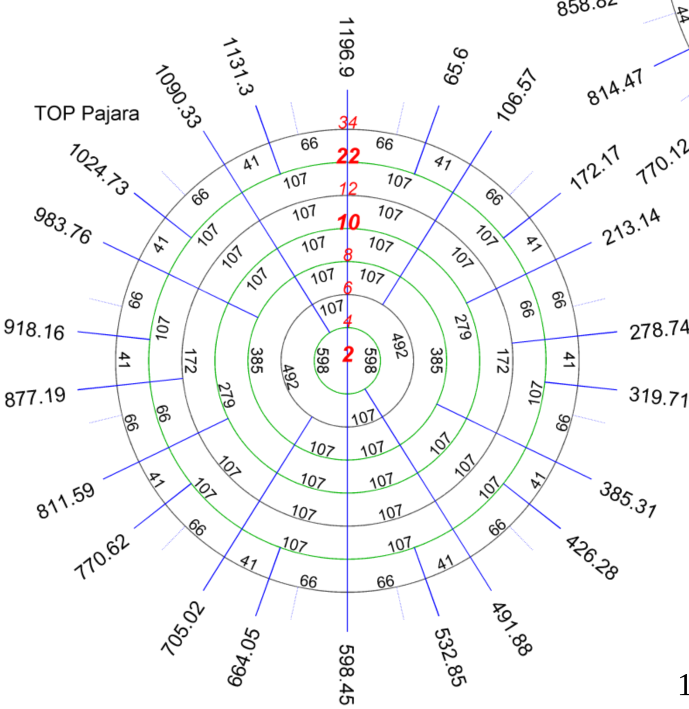
TOP Nautilus
1202.66
41.08
82.97
124.05
165.94
207.02
248.91
289.99
331.88
372.96
414.85
455.93
497.82
538.9
580.79
621.87
663.76
704.84
746.73
787.81
829.7
870.78
912.67
953.75
995.64
1036.72
1078.61
1119.69
1161.58
43
29
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
1203
1120
1037
954
871
788
705
622
539
456
373
290
207
124
41
41

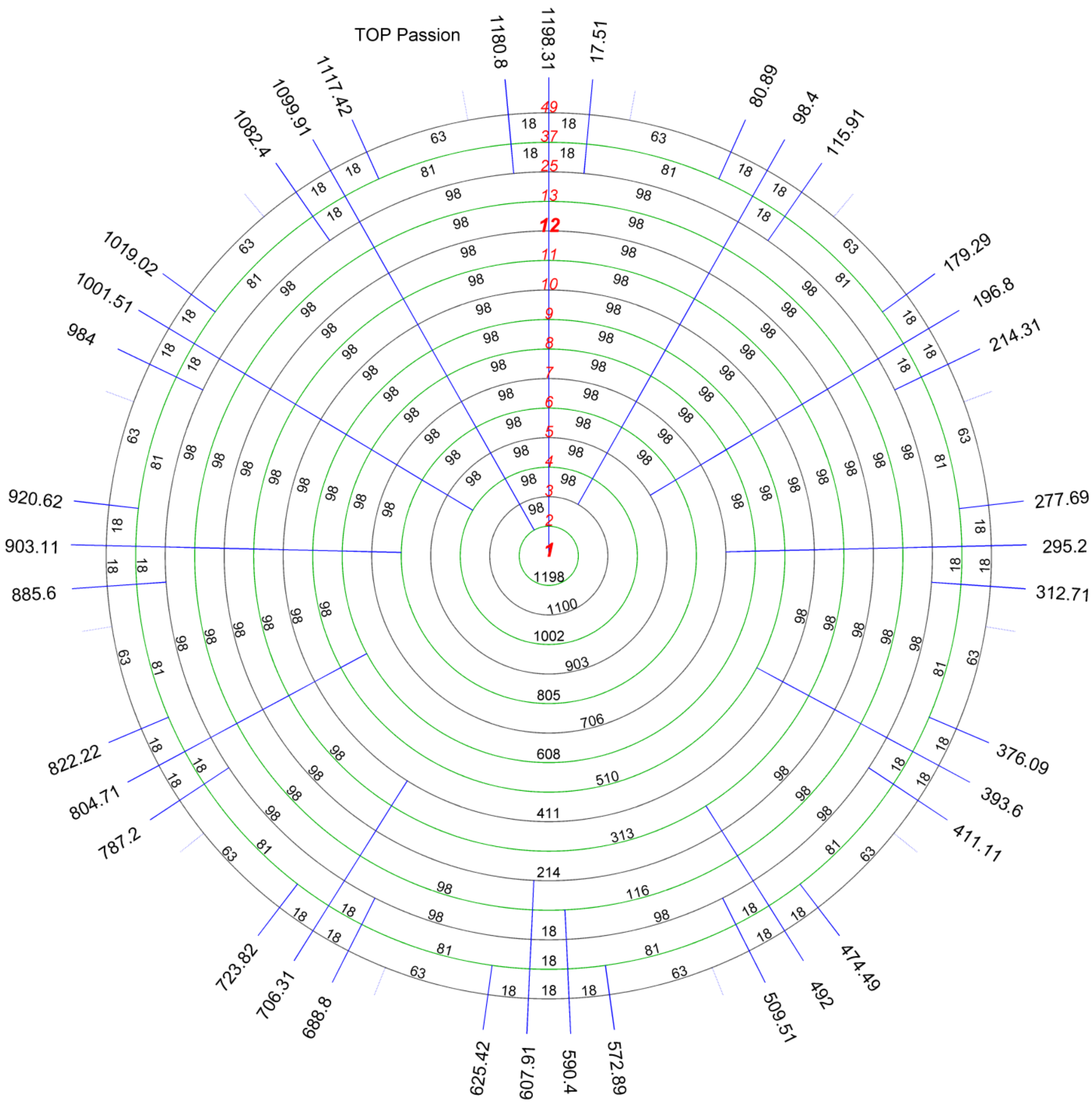

TOP Negripent
1201.82
66.56
126.14
192.7
252.28
318.84
378.42
444.98
504.56
571.12
630.7
697.26
756.84
823.4
882.98
949.54
1009.12
1075.68
1135.26
1202
1076
950
823
697
571
445
319
193

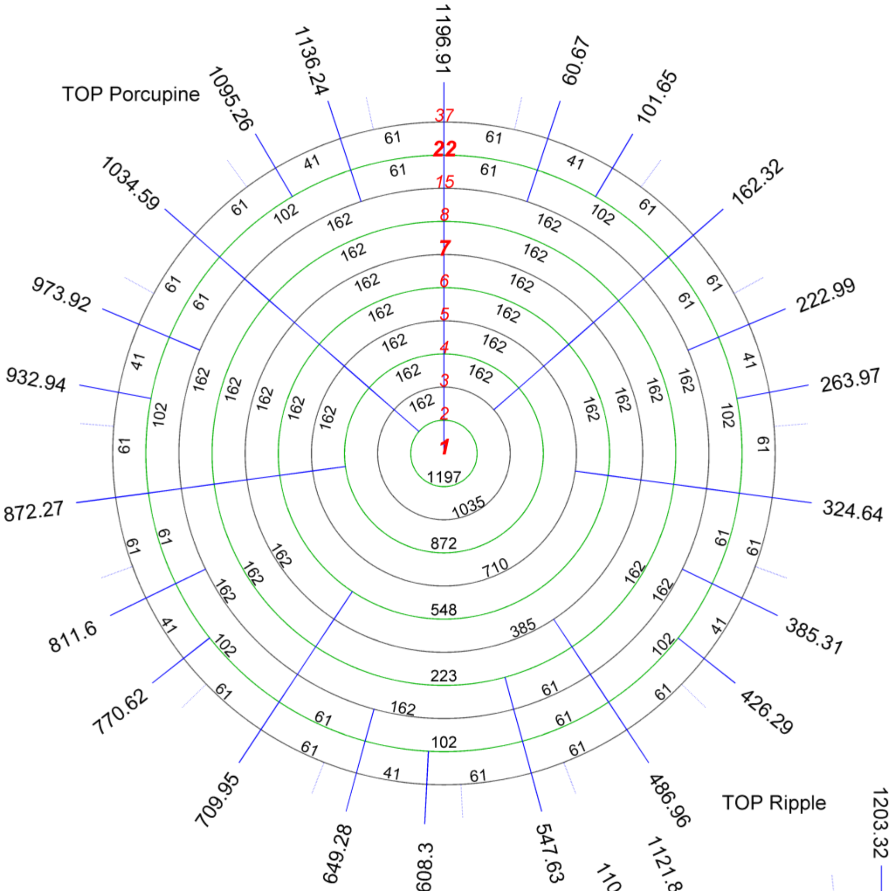

TOP Negrisept
1203.19
79.63
124.84
204.47
249.68
329.31
374.52
454.15
499.36
578.99
624.2
703.83
749.04
828.67
873.88
953.51
998.72
1078.35
1123.56
1203
1078
954
829
704
579
454
329
204

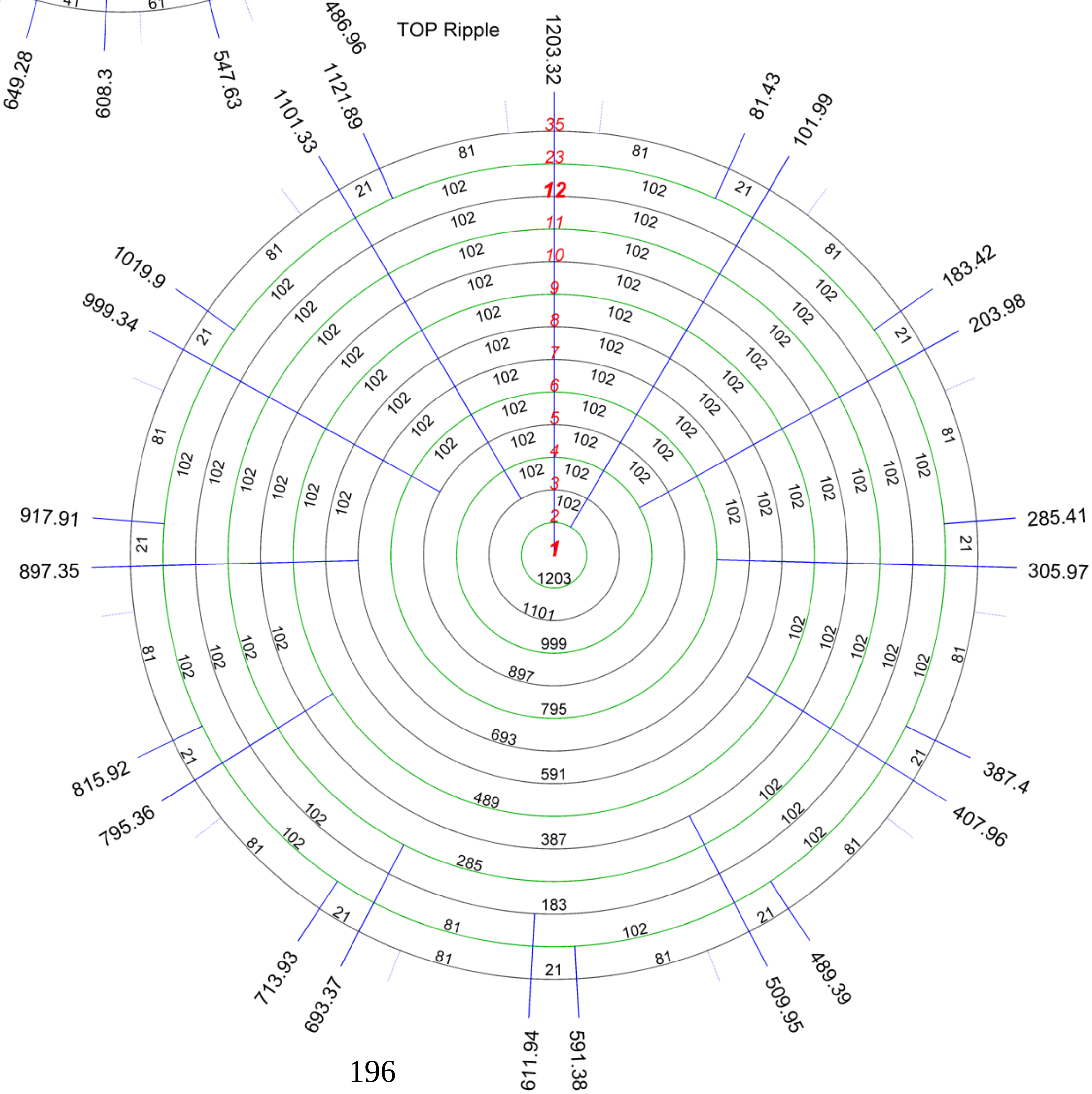

TOP Orson
1200.24
44.37
69.27
113.64
158.01
202.38
227.28
271.65
316.02
340.92
385.29
429.66
474.03
498.93
543.3
587.67
612.57
656.94
701.31
726.21
770.58
814.95
859.32
884.22
928.59
972.96
997.86
1042.23
1086.6
1130.97
1155.87


TOP Orwell
1199.53
44.35
69.22
113.57
157.92
202.27
227.14
271.49
315.84
340.71
385.06
429.41
473.76
498.63
542.98
587.33
612.2
656.55
700.9
725.77
770.12
814.47
858.82
883.69
928.04
972.39
997.26
1041.61
1085.96
1130.31
1155.18

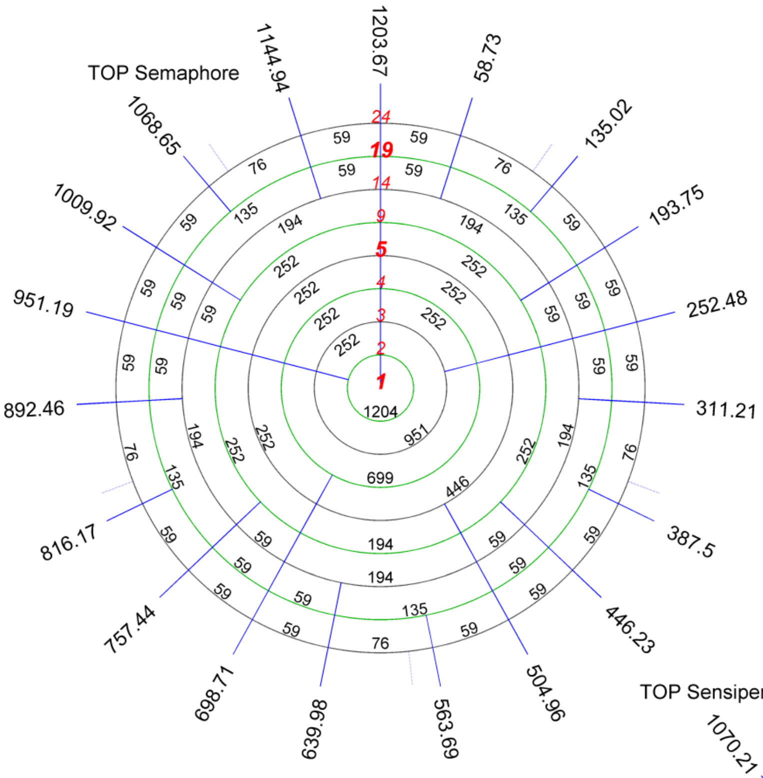

TOP Pajara
1196.9
65.6
106.57
172.17
213.14
278.74
319.71
385.31
426.28
491.88
532.85
598.45
664.05
705.02
770.62
811.59
877.19
918.16
983.76
1024.73
1090.33
1131.3


TOP Passion
1198.31
17.51
80.89
98.4
115.91
179.29
196.8
214.31
277.69
295.2
312.71
376.09
393.6
411.11
474.49
492
509.51
572.89
590.4
607.91
625.42
688.8
706.31
723.82
787.2
804.71
822.22
885.6
903.11
920.62
984
1001.51
1019.02
1082.4
1099.91
1117.42
1180.8
1
2
3
4
5
6
7
8
9
10
11
12
13
25
37
49
1198
1100
1002
903
805
706
608
510
411
313
214
116
18
63
81
98

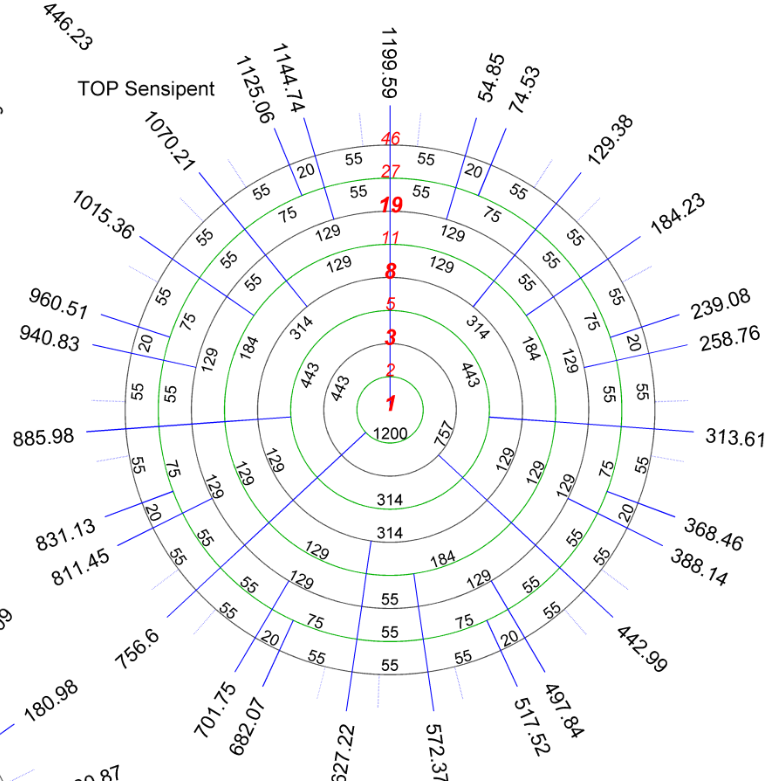

TOP Porcupine
1196.91
60.67
101.65
162.32
222.99
263.97
324.64
385.31
426.29
486.96
547.63
608.3
649.28
709.95
770.62
811.6
872.27
932.94
973.92
1034.59
1095.26
1136.24


TOP Ripple
1203.32
81.43
101.99
183.42
203.98
285.41
305.97
387.4
407.96
489.39
509.95
591.38
611.94
693.37
713.93
795.36
815.92
897.35
917.91
999.34
1019.9
1101.33
1121.89

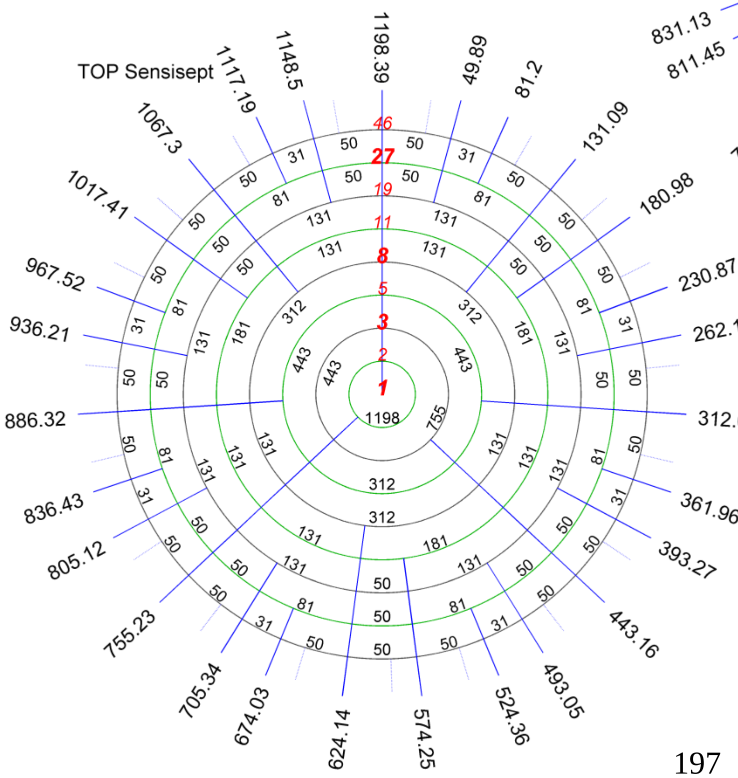
TOP Semaphore

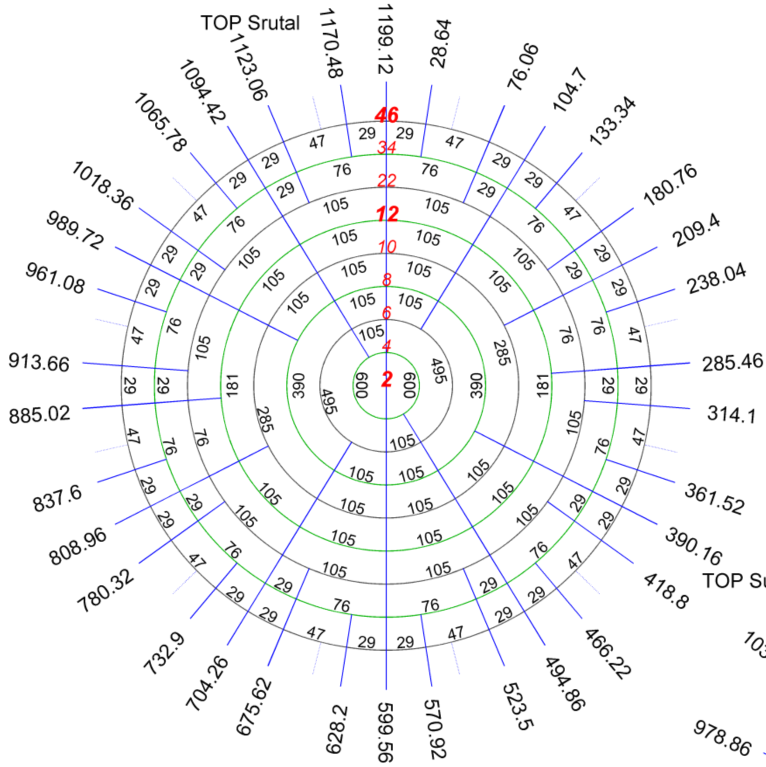


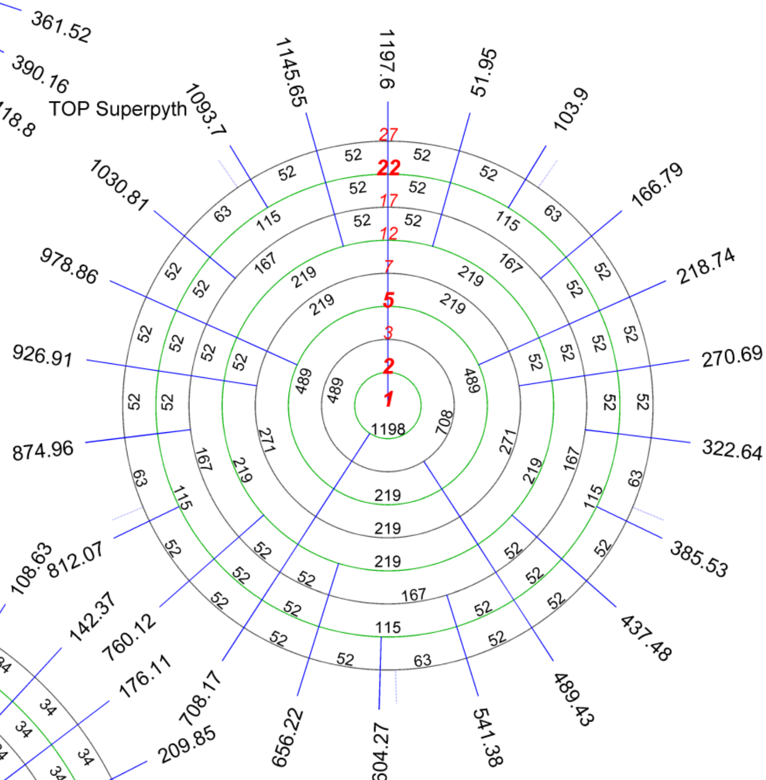
TOP Sensipent

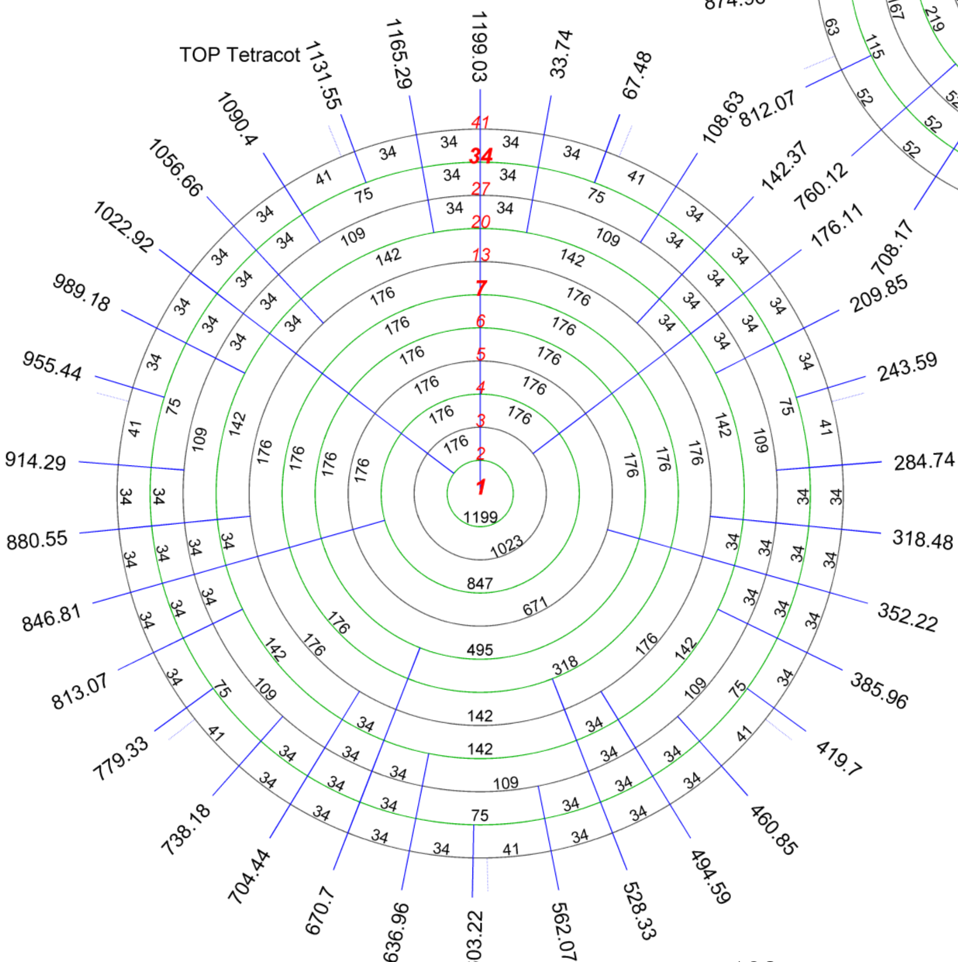


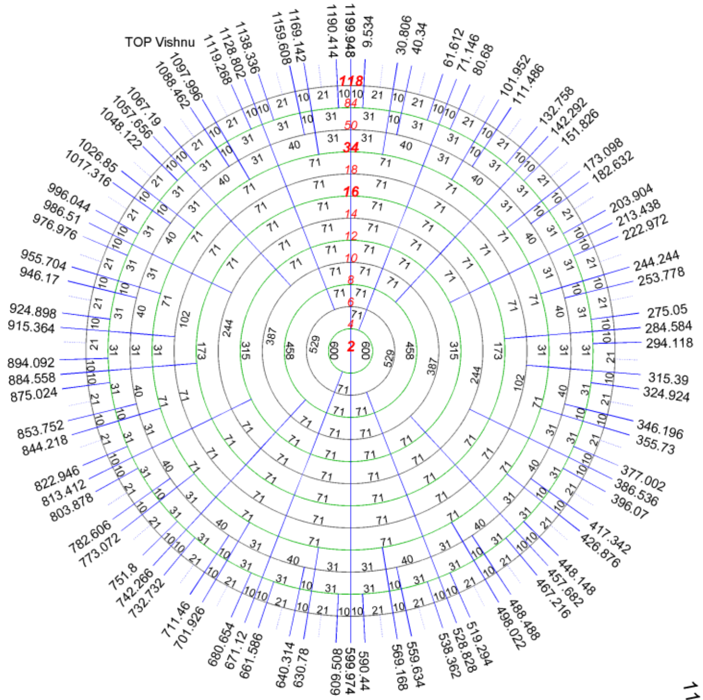
TOP Sensisept




TOP Srutal
1199.12
28.64
76.06
104.7
133.34
180.76
209.4
238.04
285.46
314.1
361.52
390.16
418.8
466.22
494.86
523.5
570.92
599.56
628.2
675.62
704.26
732.9
780.32
808.96
837.6
885.02
913.66
961.08
989.72
1018.36
1065.78
1094.42
1123.06
1170.48


TOP Superpyth
1197.6
51.95
103.9
166.79
218.74
270.69
322.64
385.53
437.48
489.43
541.38
604.27
656.22
708.17
760.12
812.07
874.96
926.91
978.86
1030.81
1093.7
1145.65


TOP Tetracot
1199.03
33.74
67.48
108.63
142.37
176.11
209.85
243.59
284.74
318.48
352.22
385.96
419.7
460.85
494.59
528.33
562.07
603.22
636.96
670.7
704.44
738.18
779.33
813.07
846.81
880.55
914.29
955.44
989.18
1022.92
1056.66
1090.4
1131.55
1165.29



Originally published in
Xenharmonikôn, An Informal Journal of Experimental Music, Volume 18 (Summer 2006).

This high resolution, searchable and extractable PDF version was produced by Carl Lumma and Dave Keenan, 12/28/2015